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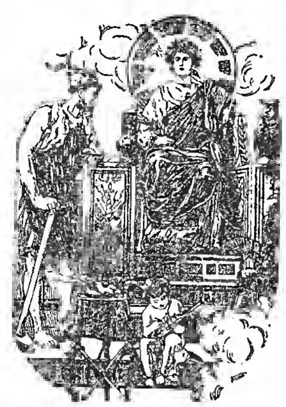

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PHILOSOPHICAL

TRANSACTIONS,

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCXVI.

PART I.

LONDON,

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ADVERTISEMENT.

THE Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions,* take this opportunity to acquaint the Public, that it fully appears, as well from the council-books and journals of the Society, as from repeated declarations which have been made in several former *Transactions,* that the printing of them was always, from time to time, the single act of the respective Secretaries, till the Forty-seventh Volume: the Society, as a Body, never interesting themselves any further in their publication, than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the Public, that their usual meetings were then continued, for the improvement of knowledge, and benefit of mankind, the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed, to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to

be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings, contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they receive them, are to be considered in no other light than as a matter of civility, in return for the respect shewn to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities, of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public news-papers, that they have met with the highest applause and approbation. And therefore it is hoped, that no regard will hereafter be paid to such reports, and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

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APPENDIX.

Meteorological Journal kept at the Apartments of the Royal Society, by Order of the President and Council.

The PRESIDENT and COUNCIL of the ROYAL SOCIETY adjudged the Medal on Sir GODFREY COPLEY's Donation, for the year 1815, to DAVID BREWSTER, LL.D., for his Paper on the Polarisation of Light by Reflection, printed in the Philosophical Transactions; And the Gold and Silver Medals on COUNT RUMFORD's Donation, to WILLIAM CHARLES WELLS, M.D. for his Essay on Dew, published in the course of the preceding year.

PHILOSOPHICAL

TRANSACTIONS.

I. *On the fire-damp of coal mines, and on methods of lighting the mines so as to prevent its explosion. By Sir* H. Davy, *LL. D. F. R. S. V. P. R. I.*

Read November 9, 1815.

THE accidents arising from the explosion of the fire-damp or inflammable gas of coal mines, mixed with atmospherical air, are annually becoming more frequent and more destructive in the collieries in the North of England.

A committee has been for some time formed at Sunderland for the benevolent purpose of investigating the causes of these accidents, and of searching for means of preventing them. In consequence of an invitation from the Rev. Dr. Gray, one of the most active members of this committee, I was induced to turn my attention to the subject. I went to the North of England, and visited some of the principal collieries in the neighbourhood of Newcastle, for the purpose of ascertaining the condition of the workings, and the state of their ventilation. I found the greatest desire to assist my enquiries in the gentlemen acquainted with the northern collieries, as well as in the inspectors or viewers of the mines;

and I have particular obligations on this point to the Rev. Dr. GRAY, CUTHBERT ELLISON, Esq. M. P., the Rev. JOHN HODGSON, Mr. BUDDLE, and Mr. DUNN. Dr. FENWICK, Dr. CLANNY, and Mr. FENWICK, likewise kindly offered me their assistance.

From the information which I collected on the spot, increased by the perusal of a report of Mr. BUDDLE on the state of the mines, I was convinced that, as far as ventilation was concerned, the resources of modern science had been fully employed; and that a mode of preventing accidents was only to be sought for, in a method of lighting the mines free from danger, and which, by indicating the state of the air in the part of the mine where inflammable air was disengaged, so as to render the atmosphere explosive, should oblige the miners to retire till the workings were properly cleared.

An account of an ingenious apparatus for burning a candle supplied with atmospherical air by a bellows through water, has been published in the Philosophical Transactions, by Dr. CLANNY; but I believe this apparatus has not yet been used in any of the collieries.

The common means employed for lighting those parts of the mine where danger is apprehended from the fire-damp, is by a steel wheel, which, being made to revolve in contact with flint, affords a succession of sparks: but this apparatus always requires a person to work it; and, though much less liable to explode the fire-damp than a common candle, yet it is said to be not entirely free from danger.

Mr. BUDDLE having obligingly shown to me the degree of light required for working the collieries, I made several ex-

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periments, with the hope of producing such a degree of light, without active inflammation; I tried KUNCKEL's, CANTON's, and BALDWIN's phosphorus, and likewise the electrical light in close vessels, but without success; and even had these degrees of light been sufficient, the processes for obtaining them, I found, would be too complicated and difficult for the miners.

The fire-damp has been shown by Dr. HENRY, in a very ingenious paper published in the nineteenth volume of Nicholson's Journal, to be light carburetted hydrogene gas, and Dr. THOMSON has made some experiments upon it; but the degree of its combustibility, as compared with that of other inflammable gases, has not, I believe, been examined, nor have many different specimens of it been analysed; and it appeared to me, that some minute chemical experiments on its properties ought to be the preliminary steps to enquiries respecting methods of preventing its explosion. I therefore procured various specimens of the fire-damp in its purest state, and made a number of experiments upon it. And in examining its relations to combustion I was so fortunate as to discover some properties belonging to it, which appear to lead to very simple methods of lighting the mines, without danger to the miners, and which, I hope, will supply the desideratum so long anxiously required by humanity. I shall in the following pages have the honour of describing these properties, and the methods founded upon them, to the Royal Society, and I shall conclude with some general observations.

The fire-damp is produced in small quantities in coal mines, during the common process of working.

The Rev. Mr. HODGSON informed me, that on pounding some common Newcastle coal fresh from the mine in a cask furnished with a small aperture, the gas from the aperture was inflammable. And on breaking some large lumps of coal under water, I ascertained that they gave off inflammable gas.* Gas is likewise disengaged from bituminous shist, when it is worked.

The great sources of the fire-damp in mines are, however, what are called blowers, or fissures in the broken strata, near dykes, from which currents of fire-damp issue in considerable quantity, and sometimes for a long course of years.† When old workings are broken into, likewise, they are often found filled with fire-damp; and the deeper the mine the more common in general is this substance.

* This is probably owing to the coal strata having been formed under a pressure greater than that of the atmosphere, so that they give off elastic fluid when they are exposed to the free atmosphere: and probably coals containing animal remains, evolve not only the fire-damp, but likewise azote and carbonic acid, as in the instance of the gas sent by Dr. CLANNY.

In the Apennines, near Pietra Mala, I examined a fire produced by gaseous matter, constantly disengaged from a shist stratum: and from the results of the combustion, I have no doubt but that it was pure fire-damp. Mr. M. FARADAY, who accompanied me, and assisted me in my chemical experiments, in my journey, collected some gas from a cavity in the earth about a mile from Pietra Mala, then filled with water, and which, from the quantity of gas disengaged, is called Aqua Buja. I analysed it in the Grand Duke's laboratory at Florence, and found that it was pure light hydro-carbonate, requiring two volumes of oxygene for its combustion, and producing a volume of carbonic acid gas.

It is very probable, that these gases are disengaged from coal strata beneath the surface, or from bituminous shist above coal; and at some future period new sources of riches may be opened to Tuscany from this invaluable mineral treasure, the use of which in this country has supplied such extraordinary resources to industry.

† Sir JAMES LOWTHER found a uniform current produced in one of his mines for two years and nine months. Phil. Trans. Vol. XXXVIII. p. 112.

I have analysed several specimens of the fire-damp in the laboratory of the Royal Institution; the pure inflammable part was the same in all of them, but it was sometimes mixed with small quantities of atmospherical air, and in some instances with azote and carbonic acid.

Of 6 specimens collected by Mr. DUNN from a blower in the Hepburn Colliery, by emptying bottles of water close to it, the purest contained $\frac{1}{15}$ only of atmospherical air, with no other contamination, and the most impure contained $\frac{5}{12}$ of atmospherical air; so that this air was probably derived from the circumambient air of the mine. The weight of the purest specimen was for 100 cubical inches 19.5 grains.

One measure of it required for its complete combustion by the electric spark nearly 2 measures of oxygene, and they formed nearly 1 measure of carbonic acid.

Sulphur heated strongly, and repeatedly sublimed in a portion of it freed from oxygene by phosphorus, produced a considerable enlargement of its volume, sulphuretted hydrogene was formed, and charcoal precipitated; and it was found that the volume of the sulphuretted hydrogene produced, when it was absorbed by solution of potassa, was exactly double that of the fire-damp decomposed.

It did not act upon chlorine in the cold; but, when an electric spark was passed through a mixture of 1 part of it with 2 of chlorine, there was an explosion, with a diminution to less than $\frac{1}{4}$, and much charcoal was deposited.

The analysis of specimens of gas sent to my friend JOHN GEORGE CHILDREN, Esq. by Dr. CLANNY, afforded me similar results; but they contained variable quantities of carbonic acid gas and azote.

Different specimens of these gases were tried by the test of exposure to chlorine both in darkness and light: they exhibited no marks of the presence of olefiant gas or hydrogene; and the residuum produced by detonation with chlorine showed them to be free from carbonic oxide.

It is evident, then, that the opinion formed by other chemists respecting the fire damp is perfectly correct; and that it is the same substance as the inflammable gas of marshes, the exact chemical nature of which was first demonstrated by Mr. DALTON: and that it consists, according to my view of definite proportions, of 4 proportions of hydrogene in weight 4, and 1 proportion of charcoal in weight 11.5.

I made several experiments on the combustibility and explosive nature of the fire-damp. When 1 part of fire-damp was mixed with 1 of air, they burnt by the approach of a lighted taper, but did not explode; 2 of air and 3 of air to 1 of gas produced similar results. When 4 of air and 1 of gas were exposed to a lighted candle, the mixture being in the quantity of 6 or 7 cubical inches in a narrow necked bottle, a flame descended through the mixture, but there was no noise: 1 part of gas inflamed with 6 parts of air in a similar bottle, produced a slight whistling sound: 1 part of gas with 8 parts of air, rather a louder sound: 1 part with 10, 11, 12, 13 and 14 parts, still inflamed, but the violence of combustion diminished. In 1 part of gas and 15 parts of air, the candle burnt without explosion with a greatly enlarged flame; and the effect of enlarging the flame, but in a gradually diminishing ratio, was produced as far as 30 parts of air to 1 of gas.

The mixture which seemed to possess the greatest explo-

sive power, was that of 7 or 8 parts of air to 1 of gas; but the report produced by 50 cubical inches of this mixture was less than that produced by $\frac{1}{10}$ of the quantity of a mixture of 2 parts of atmospherical air and 1 of hydrogene.

It was very important to ascertain the degree of heat required to explode the fire-damp mixed with its proper proportion of air.

I found that a common electrical spark would not explode 5 parts of air and 1 of fire-damp, though it exploded 6 parts of air and 1 of damp: but very strong sparks from the discharge of a Leyden jar, seemed to have the same power of exploding different mixtures of the gas as the flame of the taper. Well burned charcoal, ignited to the strongest red heat, did not explode any mixture of air and of the fire-damp; and a fire made of well burned charcoal, i. e. charcoal that burned without flame, was blown up to whiteness by an explosive mixture containing the fire-damp, without producing its inflammation. An iron rod at the highest degree of red heat, and at the common degree of white heat, did not inflame explosive mixtures of the fire-damp; but, when in brilliant combustion, it produced the effect.

The flame of gaseous oxide of carbon as well as of olefiant gas exploded the mixtures of the fire-damp.

In respect of combustibility, then, the fire-damp differs most materially from the other common inflammable gases. Olefiant gas, which I have found explodes mixed in the same proportion with air, is fired by both charcoal and iron heated to dull redness. Gaseous oxide of carbon, which explodes when mixed with 2 parts of air, is likewise inflammable by red hot iron and charcoal. And hydrogene, which explodes when mixed with $\frac{3}{7}$ of its volume of air, takes fire at the lowest

visible heat of iron and charcoal; and the case is the same with sulphuretted hydrogene.

I endeavoured to ascertain the degree of expansion of mixtures of fire-damp and air during their explosion, and likewise their power of communicating flame through apertures to other explosive mixtures.

I found that when 6 of air and 1 of fire-damp were exploded over water by a strong electrical spark, the explosion was not very strong, and, at the moment of the greatest expansion, the volume of the gas did not appear to be increased more than $\frac{1}{2}$.

In exploding a mixture of 1 part of gas from the distillation of coal, and 8 parts of air in a tube of a quarter of an inch in diameter and a foot long, more than a second was required before the flame reached from one end of the tube to the other; and I could not make any mixture explode in a glass tube $\frac{1}{7}$ of an inch in diameter: and this gas was more inflammable than the fire-damp, as it consisted of carburetted hydrogene gas mixed with some olefiant gas.

In exploding mixtures of fire-damp and air in a jar connected with the atmosphere by an aperture of half an inch, and connected with a bladder by a stopcock, having an aperture of about $\frac{1}{6}$ of an inch,* I found that the flame passed into the atmosphere, but did not communicate through the stopcock, so as to inflame the mixture in the bladder: and in comparing the power of tubes of metal and those of glass, it appeared that the flame passed more readily through glass tubes of the same diameter; and that explosions were stopped

* Since these experiments were made, Dr. WOLLASTON has informed me, that he and Mr. TENNANT had observed some time ago, that mixtures of the gas from the distillation of coal and air, would not explode in very small tubes.

by metallic tubes of $\frac{1}{5}$ of an inch,* when they were $1\frac{1}{2}$ inch long; and this phenomenon probably depends upon the heat lost during the explosion in contact with so great a cooling surface, which brings the temperature of the first portions exploded below that required for the firing of the other portions. Metal is a better conductor of heat than glass: and it has been already shown that the fire-damp requires a very strong heat for its inflammation.

Mixture of the gas with air I found, likewise, would not explode in metallic canals or troughs, when their diameter was less than the $\frac{1}{7}$ of an inch, and their depth considerable in proportion to their diameter; nor could explosions be made to pass through such canals.

Explosions likewise I found would not pass through very fine wire sieves or wire gauze.

I mixed azote and carbonic acid in different quantities with explosive mixtures of fire-damp, and I found that even in very small proportions they diminished the velocity of the inflammation. Azote, when mixed in the proportion of 1 to 6 of an explosive mixture, containing 12 of air and 1 of fire-damp, deprived it of its power of explosion; when 1 part of azote was mixed with 7 of an explosive mixture, only a feeble blue flame slowly passed through the mixture.

1 part of carbonic acid to 7 of an explosive mixture deprived it of the power of exploding; so that its effects are more remarkable than those of azote; probably, in consequence of its greater capacity for heat, and probably, likewise,

* I do not give this result as perfectly exact, as the bore of the metallic tube had not the same polish as that of the tube of glass.

of a higher conducting power connected with its greater density.

The consideration of these various facts, led me to adopt a form of a lamp, in which the flame, by being supplied with only a limited quantity of air, should produce such a quantity of azote and carbonic acid, as to prevent the explosion of the fire-damp, and which, by the nature of its apertures for giving admittance and exit to the air, should be rendered incapable of communicating any explosion to the external air.

If in a close lantern, supplied with a small aperture below and another above, a lighted lamp having a very small wick be placed, the natural flame gradually diminishes, till it arrives at a point at which the supply of air is sufficient for the combustion of a certain small quantity of oil; if a lighted taper be introduced into the lantern through a small door in the side, which is instantly closed, both lights will burn for a few seconds, and be extinguished together.

A similar phenomenon occurs, if, in a close lantern, supplied with a quantity of air merely sufficient to support a certain flame, a mixture of fire-damp and air is gradually admitted, the first effect of the fire-damp is to produce a larger flame round that of the lamp, and this flame, consuming the oxygene which ought to be supplied to the flame of the lamp, and the standard of the power of the air to support flame being lowered by the admixture of fire-damp and by its rarefaction, both the flame of the fire-damp and that of the taper are extinguished together; and as the air contained a certain quantity of azote and carbonic acid before the admission of the fire-damp, their effect, by mixing with it, is such as to prevent an explosion in any part of the lantern.

I tried several experiments on the burning of a flame in atmospheres containing fire-damp. I inclosed a taper in a little close lantern, having a small aperture below and a larger one above, of such size that the taper burnt with a flame a little below its natural size. I placed this lantern, the taper being lighted, on a stand under a large glass receiver standing in water, having a curved tube containing a little water, adapted to its top to confine the air, and which was of such a capacity as to enable the candle to burn for some minutes; I then rapidly threw a quantity of fire-damp into the receiver from a bladder, so as to make the atmosphere in it explosive. As the fire-damp mixed with the air, the flame of the taper gradually enlarged, till it half filled the lantern; it then rapidly diminished, and was suddenly extinguished without the slightest explosion. I examined the air of the receiver after the experiment, and found it highly explosive.

I tried similar experiments, throwing in mixtures of air and fire-damp, some containing the maximum, and others the minimum of fire-damp necessary for explosion, and always with the same satisfactory results. The flame considerably increased, and was soon extinguished.

I introduced a lighted lantern to which air was supplied by two glass tubes of $\frac{1}{10}$ of an inch in diameter and half an inch long, into a large jar containing an explosive mixture of 1 part of fire-damp and 10 parts of air; the taper burnt at first with a feeble light, the flame soon became enlarged, and was then extinguished. I repeated these experiments several times, and with a perfect constancy of result.

It is evident, then, that to prevent explosions in coal mines,

it is only necessary to use air-tight lanterns, supplied with air from tubes or canals of small diameter, or from apertures covered with wire gauze placed below the flame, through which explosions cannot be communicated, and having a chimney at the upper part, on a similar system for carrying off the foul air; and common lanterns may be easily adapted to the purpose, by being made air-tight in the door and sides, by being furnished with the chimney, and the system of safety apertures below and above.

The principle being known, it is easy to adopt, and multiply practical applications of it.

The first safe lantern that I had constructed, was made of tin-plate, and the light emitted through four glass plates in the sides. The air was admitted round the bottom of the flame from a number of metallic tubes of $\frac{1}{8}$ of an inch in diameter, and an inch and $\frac{1}{2}$ long. The chimney was composed of two open cones, having a common base perforated with many small apertures, and fastened to the top of the lantern, which was made tight in a pneumatic rim containing a little oil; the upper and lower apertures in the chimney were about $\frac{1}{3}$ of an inch: the lamp, which was fed with oil, gave a steady flame of about an inch high and half an inch in diameter. When the lantern was slowly moved, the lamp continued to burn, but more feebly, and when it was rapidly moved, it went out. To obviate this circumstance, I surrounded the bottom of the lantern with a perforated rim; and this arrangement perfectly answered the end proposed.

I had another chimney fitted to this lantern, furnished with a number of safety tin-plate tubes of the sixth of an inch in diameter and two inches long: but they diminished consi-

derably the size of the flame, and rendered it more liable to go out by motion; and the following experiments appear to show, that if the diameter of the upper orifice of the chimney be not very large, it is scarcely possible that any explosion produced by the flame can reach it.

I threw into the safe lantern with the common chimney, a mixture of 15 parts of air and 1 of fire-damp: the flame was immediately greatly enlarged, and the flame of the wick seemed to be lost in the larger flame of the fire-damp. I placed a lighted taper above the orifice of the chimney: it was immediately extinguished, but without the slightest previous increase of its flame, and even the wick instantly lost its fire by being plunged into the chimney.

I introduced a lighted taper into a close vessel containing 15 parts of air and 1 of gas from the distillation of coal, suffered it to burn out in the vessel, and then analyzed the gas. After the carbonic acid was separated, it appeared by the test of nitrous gas to contain nearly $\frac{1}{3}$ of of its original quantity of oxygene; but detonation with a mixture of equal parts of hydrogene and oxygene proved that it contained no sensible quantity of carburetted hydrogene gas.

It is evident, then, that when in the safe lantern, the air gradually becomes contaminated with fire-damp, this fire-damp will be consumed in the body of the lantern; and that the air passing through the chimney, cannot contain any inflammable mixture.

I made a direct experiment on this point. I gradually threw an explosive mixture of fire-damp and air into the safe lantern from a bladder furnished with a tube which opened by a large aperture above the flame; the flame became enlarged, and

by a rapid jet of gas I produced an explosion in the body of the lantern ; there was not even a current of air through the safety tubes at the moment, and the flame did not appear to reach above the lower aperture of the chimney ; and the explosion merely threw out from it a gust of foul air.

The second safety lantern that I have had made is upon the same principle as the first, except that instead of tubes, *safety canals* are used, which consist of close concentric hollow metallic cylinders of different diameters, and placed together so as to form circular canals of the diameter of from $\frac{1}{25}$ to $\frac{1}{40}$ of an inch, and an inch and $\frac{7}{10}$ long, by which air is admitted in much larger quantities than by the small tubes. In this arrangement there is so free a circulation of air, that the chimney likewise may be furnished with safety canals.

I have had lamps made for this kind of lantern which stand on the outside, and which may be supplied with oil and cotton without any necessity of opening the lantern; and in this case the chimney is soldered to the top, and the lamp is screwed into the bottom, and the wick rises above the air canals.

I have likewise had glass lamps with a single wick, and argand lamps made on the same principle, the chimney being of glass covered with a metallic top containing the safety canals, and the air entering close to the flame through the circular canals.

The third kind of safe lamp or lantern, and which is by far the most simple, is a close lamp or lantern into which the air is admitted, and from which it passes, through apertures covered with *brass wire gauze* of $\frac{1}{200}$ of an inch in thickness, the apertures of which should not be more than $\frac{1}{120}$ of an

inch; this stops explosions as well as long tubes or canals, and yet admits of a free draught of air.

Having succeeded in the construction of safe lanterns and lamps, equally portable with common lanterns and lamps, which afforded sufficient light, and which bore motion perfectly well, I submitted them individually to practical tests, by throwing into them explosive atmospheres of fire-damp and air. By the natural action of the flame drawing air through the air canals, from the explosive atmosphere, the light was uniformly extinguished; and when an explosive mixture was forcibly pressed into the body of the lamp, the explosion was always stopped by the safety apertures, which may be said figuratively to act as a sort of *chemical fire sieves* in separating flame from air. But I was not contented with these trials, and I submitted the safe canals, tubes, and wire gauze fire sieves, to much more severe tests: I made them the medium of communication between a large glass vessel filled with the strongest explosive mixture of carburetted hydrogene and air, and a bladder $\frac{2}{3}$ or $\frac{1}{2}$ full of the same mixture, both insulated from the atmosphere. By means of wires passing near the stop-cock of the glass vessel, I fired the explosive mixture in it by the discharge of a Leyden jar. The bladder always expanded at the moment the explosion was made; a contraction as rapidly took place; and a lambent flame played round the mouths of the safety apertures, open in the glass vessel; but the mixture in the bladder did not explode: and by pressing some of it into the glass vessel, so as to make it replace the foul air, and subjecting it to the electric spark, repeated explosions were produced, proving the perfect security of the safety apertures; even when acted

on by a much more powerful explosion than could possibly occur from the introduction of air from the mines.

These experiments held good whatever was the proportions of the explosive mixture and whatever was the size of the glass vessel, (no one was ever used containing more than a quart) provided as many as 12 metallic tubes were used of $\frac{1}{7}$ of an inch in diameter, and $2\frac{1}{2}$ inches long; or provided the circular metallic canals, were $\frac{1}{25}$ of an inch in diameter, $1.\frac{1}{7}$ of an inch deep, and at least 2 inches in circumference; or provided the wire gauze had apertures of only $\frac{1}{120}$ of an inch. When 12 metallic tubes were employed as the medium of communication, $\frac{1}{7}$ of an inch in diameter and an inch long, the explosion was communicated by them into the bladder. Four glass tubes of the $\frac{1}{16}$ of an inch in diameter and 2 inches long, did not communicate the explosion; but *one* of this diameter and length produced the effect. The explosion was stopped by a single tube $\frac{1}{28}$* of an inch in diameter, when it was 3 inches long, but not when it was 2 inches long.

The explosion was stopped by the metallic gauze of $\frac{1}{120}$ when it was placed between the exploding vessel and the bladder, though it did not present a surface of more than half a square inch, and the explosive mixture in the bladder in passing through it to supply the vacuum produced in the glass vessel, burnt on the surface exposed to the glass vessel for some seconds, producing a murmuring noise.

A circular canal $\frac{1}{25}$ of an inch in diameter, an inch and a

* These results appear at first view contradictory to those mentioned page 9. But it must be kept in view that the first set of experiments were made in tubes open in the air, and the last in tubes exposed to the whole force of air explosion, and connected only with close vessels filled with explosive mixtures.

half in circumference, and $1\frac{7}{10}$ of an inch deep, communicated explosion, but four concentric canals, of the same depth and diameter, and of which the smallest was two inches in diameter, and separated from each other only by their sides, which were of brass, and about $\frac{1}{40}$ of an inch in thickness, did not suffer the explosion to act through them.

It would appear then, that the smaller the circumference of the canal, that is the nearer it approaches to a tube, the greater must be its depth, or the less its diameter to render it safe.

I did not perceive any difference in these experiments, when the metals of the apertures were warmed by repeated explosions; it is probable, however, that considerable elevation of temperature would increase the power of the aperture to pass the explosion; but the difference between the temperature of flame, and that marked on our common mercurial scale, is so great that the addition of a few degrees of heat probably does not diminish perceptibly the cooling power of a metallic surface, with regard to flame.

By diminishing the diameter of the air canals, their power of passing the explosion is so much diminished that their depth and circumference may be brought extremely low. I found that flame would not pass through a canal of the $\frac{1}{70}$ of an inch in diameter, when it was $\frac{1}{4}$ of an inch deep, and forming a cylinder of only $\frac{1}{4}$ of an inch in circumference; and a number of apertures of $\frac{1}{100}$ of an inch are safe when their depth is equal to their diameter. It is evident from these facts, that metallic doors, or joinings in lamps, may be easily made safe by causing them to project upon and fit closely to parallel metallic surfaces.

Longitudinal air canals of metal may, I find, be employed with the same security as circular canals; and a few pieces of tin-plate soldered together with wires to regulate the diameter of the canal, answer the purpose of the feeder or safe chimney, as well as drawn cylinders of brass.

A candle will burn in a lantern or glass tube made safe with metallic gauze, as well as in the open air; I conceive, however, that oil lamps, in which the wick will always stand at the same height, will be preferred.

But the principle applies to every kind of light, and its entire safety is demonstrated.

When the fire-damp is so mixed with the external atmosphere as to render it explosive, the light in the safe lantern or lamp will be extinguished, and warning will be given to the miners to withdraw from, and to ventilate, that part of the mine.

If it be necessary to be in a part of the mine where the fire-damp is explosive, for the purpose of clearing the workings, taking away pillars of coal, or other objects, the workmen may be lighted by a fire made of charcoal, which burns without flame, or by the steel mill, though this does not afford such entire security from danger as the charcoal fire.

It is probable, that when explosions occur from the sparks from the steel mill, the mixture of the fire-damp is in the proportion required to consume all the oxygene of the air, for it is only in about this proportion that explosive mixtures can be fired by electrical sparks from a common machine.

As the wick may be moved without communication between the air in the safe lantern or lamp and the atmosphere, there is no danger in trimming or feeding them; but they should be

lighted in a part of the mine where there is no fire-damp, and by a person charged with the care of the lights: and by these inventions, used with such simple precautions, there is every reason to believe a number of lives will be saved, and much misery prevented. Where candles are employed in the open air in the mines, life is extinguished by the explosion; with the safe lantern or safe lamp the light is only put out, and no other inconvenience will occur.

It does not appear, by what I have learnt from the miners, that breathing an atmosphere containing a certain mixture of fire-damp near or even at the explosive point, is attended with any bad consequence. I ascertained that a bird lived in a mixture of equal parts of fire-damp and air; but he soon began to show symptoms of suffering. I found a slight head ache produced by breathing for a few minutes an explosive mixture of fire-damp and air: and if merely the health of the miners be considered, the fire-damp ought always to be kept far below the point of its explosive mixture.

Miners sometimes are found alive in a mine after an explosion has taken place: this is easily explained, when it is considered that the inflammation is almost always limited to a particular spot, and that it mixes the residual air with much common air; and supposing 1 of fire-damp to 13 of air to be exploded, there will still remain nearly $\frac{1}{3}$ of the original quantity of oxygene in the residual gas: and in some experiments, made 16 years ago, I found that an animal lived, though with suffering, for a short time, in a gas containing 100 parts of azote, 14 parts of carbonic acid, and 7 parts of oxygene.

Explanation of the Plate.

Plate I.

Fig. 1. Represents the safe lantern, with its air-feeder and chimney furnished with safety metallic canals. It contains about a quart of air. The sides are of horn or glass, made air tight by putty or cement. A. is the lamp through which the circular air-feeding canals pass: they are 3 concentric hollow cylinders, distant from each other $\frac{1}{26}$ of an inch: the smallest is $2\frac{1}{2}$ inches in circumference; their depth is 2 inches. B. is the chimney, containing 4 such canals, the smallest 2 inches in circumference: above it is a hollow cylinder, with a cap to prevent dust from passing into the chimney. C. is the hole for admitting oil. D. is a long canal containing a wire by which the wick is moved or trimmed. E. is the tube forming a connection between the reservoir of oil and the chamber that supplies the wick with oil. F. is the rim round the bottom of the lantern to enable it to bear motion.

Fig. 2. Is the lamp of Fig. 1., of its natural size, the references to the letters are the same.

Fig. 3. Is a common chimney which may be used in the lantern; but the safety chimney doubles security.

Fig. 4. Exhibits the safety concentric canals or fire sieves, which if $\frac{1}{25}$ of an inch in diameter, must not be less than 2 inches in exterior circumference and 1.7 inch high.

Fig. 5. Exhibits the longitudinal safe canals or fire sieves.

Fig. 6. Exhibits a safe lamp having a glass chimney covered with tin-plate, and the safety apertures in a cylinder with a covering above: the lower part is the same as in the lantern.

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Fig. 7. An argand lamp of similar construction, with safe air canals without the flame, and metallic gauze apertures within.

Fig. 8. A tin-plate chimney top for the lamp, made safe by metallic gauze.

Fig. 9. A metallic gauze safe lamp. AAA. Screens of metallic gauze or *flame sieves*. BB. Wires for trimming the wick.

Fig. 10 A glass tube furnished with *flame sieves*, in which a common candle may be burnt. A A. The flame sieves. B. A little plate of metal to prevent the upper flame sieve from being acted on by the current of hot air.

The lamps burn brighter the higher the chimney.

From my experiments it appears, that a mere narrow throat and opening to the metallic part of the chimney, is sufficient to prevent explosions from passing through the lamp, supposing them possible; but with the safety canals or metallic gauze in the chimney the security is absolute.

The circular canals and the apertures covered with metallic gauze, are so much superior to tubes in practical application, that I have no doubt of their being generally used; I have therefore given no sketch of the first safe lantern I had constructed with tubes; but substituting tubes for canals it is exactly the same, as that represented Fig. 1.

APPENDIX.

1. IN the beginning of my inquiries I had another close lantern made, which may be called the fire-valve lantern. In this, the candle or lamp burns with its full quantity of air, admitted from an aperture below, till the air begins to be mixed with fire-damp; when, as the fire-damp increases the flame, a thermometrical spring at the top of the lantern, made of brass and steel, riveted together, and in a curved form, expands, moves a valve in the chimney, diminishes the circulation of air, and extinguishes the flame; but I did not pursue this invention, after I had discovered the properties of the fire-damp, on which the safe lantern is founded.

2. The safety of close lamps or lanterns may probably be likewise secured by sieves made of asbestus, or possibly even hair or silk, placed over the air apertures: but metallic gauze will be necessary above in the chimney. I have little doubt but that windows of fine metallic gauze may be used for giving light in lanterns, with perfect security; perhaps for the chimney it may be worth while to have fine silver plated wire gauze made.

3. The expansive powers of the fire-damp during its explosion, are so small as to render no precautions, with respect to the thickness of the glass or horn in the lamps or lanterns, necessary.

I. p. 22.

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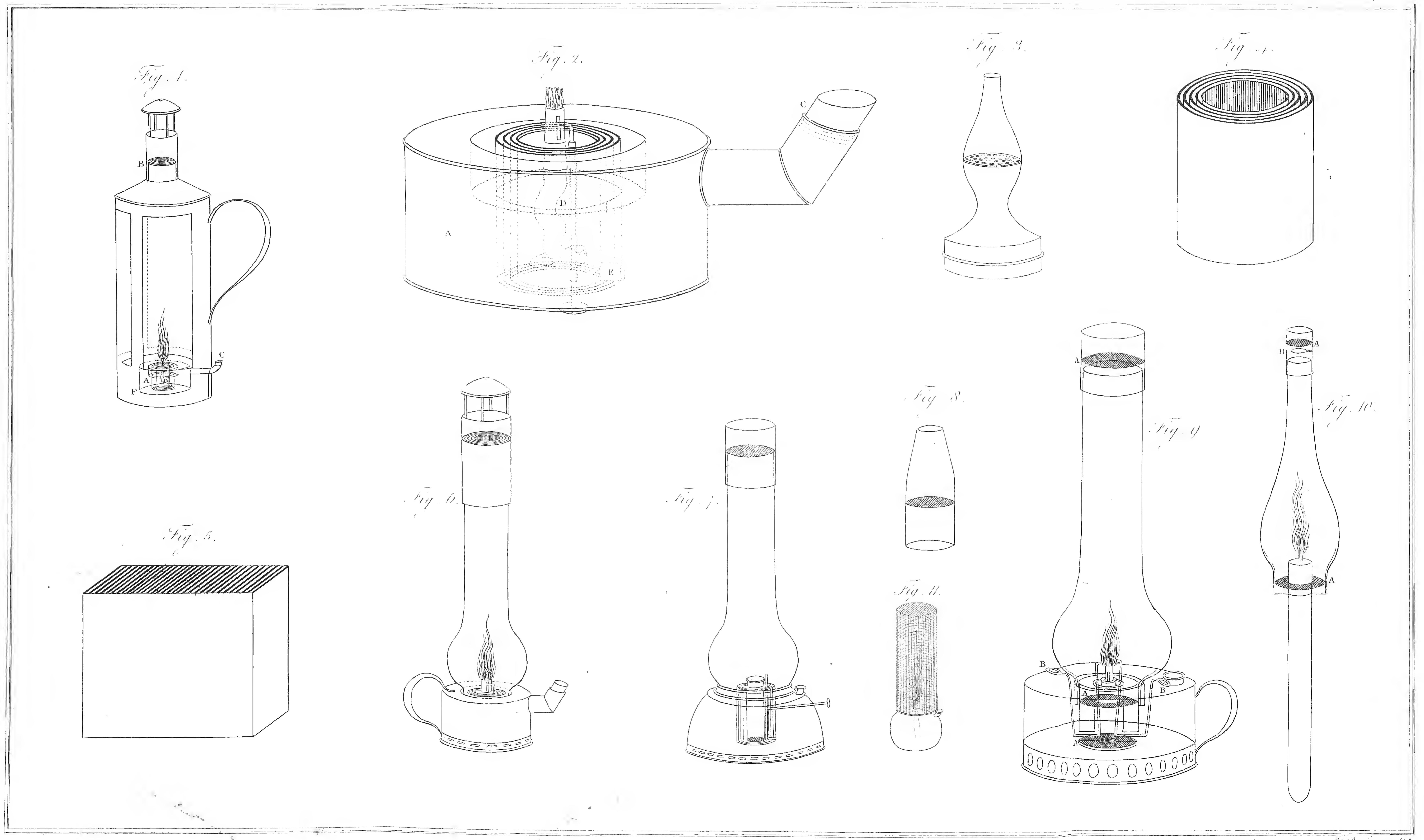

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II. *An account of an invention for giving light in explosive mixtures of fire-damp in coal mines, by consuming the fire-damp. By Sir* Humphry Davy, *LL. D. F. R. S. V. P. R. I.*

Read January 11, 1816.

I HAVE already had the honor of communicating to the Royal Society an account of a safe light, which becomes extinguished when introduced into very explosive mixtures of fire-damp; in this communication I shall describe a light which will burn in any explosive mixture of fire-damp, and the light of which arises from the combustion of the fire-damp itself.

The invention consists in covering or surrounding the flame of a lamp or candle by a wire sieve; the coarsest that I have tried with perfect safety contained 625 apertures in a square inch, and the wire was $\frac{1}{70}$ of an inch in thickness, the finest 6400 apertures in a square inch, and the wire was $\frac{1}{250}$ of an inch in diameter.

When a lighted lamp or candle screwed into a ring soldered to a cylinder of wire gauze, having no apertures, except those of the gauze or safe apertures, is introduced into the most explosive mixture of carburetted hydrogene and air, the cylinder becomes filled with a bright flame, and this flame continues to burn as long as the mixture is explosive. When the carburetted hydrogene is to the air as 1 to 12, the flame of the wick appears within the flame of the fire-damp; when the proportion is as high as 1 to 7, the flame of the wick disappears.

When the thickest wires are used in the gauze, it becomes strongly red hot, particularly at the top, but yet no explosion takes place. The flame is brighter the larger the apertures of the gauze; and the cylinder of 625 apertures to the square inch, gives a most brilliant light in a mixture of one part of gas from the distillation of coal, and 7 parts of air; the lower part of the flame is green, the middle purple, and the upper part blue.

I have tried cylinders of 6400 apertures to the square inch, in mixtures of oxygene and carburetted hydrogene, and even in mixtures of oxygene and hydrogene; and though the wire became intensely red hot, yet explosions never took place: the combustion was entirely limited to the interior of the lamp.

In all these experiments there was a noise like that produced by the burning of hydrogene gas in open tubes

These extraordinary and unexpected results lead to many enquiries respecting the nature and communication of flame; but my object, at present, is only to point out their application to the use of the collier.

All that he requires to ensure security, are small wire cages* to surround his candle or his lamp, which may be made for a few pence, and of which various modifications may be adopted; and the application of this discovery will not only preserve him from the fire-damp, but enable him to apply it to use, and to destroy it at the same time that it gives him an useful light.

* Fig. II. Pl. 1. represents this contrivance.

III. *On the developement of exponential functions; together with several new theorems relating to finite differences.* By John Frederick W. Herschel, *Esq. F. R. S.*

Read December 14, 1815.

In the year 1772, Lagrange, in a Memoir, published among those of the Berlin Academy, announced those celebrated theorems expressing the connection between simple exponential indices, and those of differentiation and integration. The demonstration of those theorems, although it escaped their illustrious discoverer, has been since accomplished by many analysts, and in a great variety of ways. Laplace set the first example in two Memoirs presented to the Academy of Sciences,* and may be supposed in the course of these researches, to have caught the first hint of the *Calcul des Fonctions Generatrices* with which they are so intimately connected; as, after an interval of two years, another demonstration of them, drawn solely from the principles of that calculus appeared, together with the calculus itself, in the memoirs of the Academy. This demonstration, involving, however, the passage from finite to infinite, is therefore (although preferable perhaps in a systematic arrangement, where all is made to flow from one fundamental principle) less elegant; not on account of any confusion of ideas, or want of evidence; but, because the ideas of finite and infinite, as such, are extraneous to symbolic language, and, if we

* Mém. des Savans Etrangers, 1773. p. 535.—Mém. de l'Acad. 1772. p. 102.

would avoid their use, much circumlocution, as well as very unwieldy formulæ must be introduced. ARBOGAST also, in his work on derivations, has given two most ingenious demonstrations of them, and added greatly to their generality; and lastly, Dr. BRINKLEY has made them the subject of a paper in the Transactions of this Society,* to which I shall have occasion again to refer. Considered as insulated truths, unconnected with any other considerable branch of analysis, the method employed by the latter author seems the most simple and elegant which could have been devised. It has however the great inconvenience of not making us acquainted with the bearings and dependencies of these important theorems, which, in this instance, as in many others, are far more valuable than the mere formulæ.

The theorems above referred to are comprehended in the equation.

$$\Delta^n u_x = \left\{ \varepsilon^{\Delta x . D} - 1 \right\}^n u_x \, ; \quad \ldots\ldots\ldots\ldots \quad (a)$$

or, more generally

$$f(1+\Delta)\, u_x = f\left\{ \varepsilon^{\Delta x . D} \right\} u_x \, ; \quad \ldots\ldots\ldots \quad (b)$$

where the Δ applies to the variation of x, and the D to the functional characteristic u; and where n may have any value whatever.

Taking these theorems for granted, I shall observe, that, in their present form, they are but abridged expressions of their meaning, and that to become practically useful, their second members must be developed in a series of the powers of $\Delta x . D$. This part of their theory has been most beautifully and satisfactorily treated by LAPLACE in the case of $n = -1$

* Phil. Trans. 1807. I.

(one of the most important). Unfortunately, his method turns upon an artifice which, although remarkably ingenious, fails to afford us any satisfaction except in this particular case; and I am not aware that his researches have since extended beyond it. The essay of Dr. BRINKLEY (the only author I have met with who has attempted the general problem) goes to the bottom of the difficulty, and leads to a formula which, considering the complex nature of the subject, must be allowed to be far more simple than could have been expected. It is often, however, advantageous to undertake the solution of the same problem by different methods. The excellent geometer I have mentioned, has adopted one which appears at first sight very inartificial. It consists in expanding the second member of the equation (a) reduced to the form

$$\left\{\frac{t}{1}+\frac{t^2}{1.2}+\frac{t^3}{1.2.3.}+\text{\&c.}\right\}^n$$

by the well-known theorem for raising a multinomial to the n^{th} power. The difficulties and apparent obstacles which this method presents, he has overcome or eluded by a singularly acute discussion of the combinations of the various numerical coefficients and their powers. But it is obvious that this method, applied to the more general equation (b), would lead into details of extreme complexity. This consideration induced me to begin with that equation, regarding the other as a particular case of it; and I have thus arrived at a general and highly interesting formula (equation (2) of the following pages) hitherto, I believe, totally unnoticed, and which in the particular case of the equation (a), when n is a positive integer, affords precisely the same result as Dr. BRINKLEY has given: when n, however, is negative, it yields an expres-

sion widely differing from his in point of form (though of course affording the same numerical results) and which in the most important case, where $n = -1$, takes a form of greater simplicity than any I am yet aware of.

I purpose then to consider the second member of (b) as developed in a series of powers of $\Delta x . D$ (which for the sake of brevity we will denote by t). If then we suppose

$$f(\varepsilon^t) = A_0 + A^1 t + A_2 t^2 + \&c.$$

we shall have

$$A_x = \frac{d^x f(\varepsilon^t)}{1.2.\ldots.x.\,dt^x};$$

where $t = o$ after the differentiations.

Now, it is easy to see that $\frac{d^x f(\varepsilon^t)}{dt^x}$ will, by performing the operations indicated, assume the form

$$K_{x,1}\,\varepsilon^t . Df(\varepsilon^t) + K_{x,2}\,\varepsilon^{2t} . D^2 f(\varepsilon^t) + \&c.$$

$K_{x,y}$ being a certain numerical coefficient, depending on an equation of differences

$$K_{x+1,y+1} = (y+1).K_{x,y+1} + K_{x,y}$$

whose complete integral is

$$K_{x,y} = C_y . y^x \quad C_{y-1} . \frac{(y-1)^x}{1} + \ldots . (-1)^{y+1} . \frac{1^x}{1.2\ldots\ldots(y-1)} . C_1$$

C_y being an arbitrary function of y, to determine which we have only to consider that $K_{x,x}$ is always, necessarily, unity; and consequently

$$C_x . x^x - C_{x-1} . \frac{(x-1)^x}{1} + \ldots\ldots\ldots (-1)^{x+1}\, C_1 . \frac{1^x}{1.2\ldots.(x-1)} = 1$$

Now, we know that

$$x^x - x . \frac{(x-1)^x}{1} + \&c. = 1.2\ldots.x$$

that is,

$$\frac{1}{1.2\ldots x}.x^x - \frac{1}{1.2\ldots(x-1)}.\frac{(x-1)^x}{1} + \&c. = 1\,;$$

whence it is plain that

$$C_y = \frac{1}{1.2\ldots\ldots y}$$

and of course that

$$K_{x,y} = \frac{y^x - \frac{y}{1}.(y-1)^x +, \&c.}{1.2\ldots\ldots y}$$

$$= \frac{\Delta^y o^x}{1.2\ldots\ldots y},$$

where $\Delta^y o^x$ denotes the first term of the y^{th} differences of the terms of a series o^x, 1^x, 2^x, &c. We have then making $t=o$,

$$\frac{d^x f(\varepsilon^t)}{dt^x} = \frac{Df(1)}{1}\Delta o^x + \ldots\ldots\ldots \frac{D^x f(1)}{1.2\ldots.x}\Delta^x o^x.$$

If we separate the symbols of operation from those of quantity, the second member of this equation may be much more elegantly written as follows:

$$\left\{\frac{D\Delta}{1} + \frac{(D\Delta)^2}{1.2} + \ldots\ldots \frac{(D\Delta)^x}{1.2\ldots.x}\right\} f(1).o^x\,; \ldots\ldots\ldots\ldots\ldots (1)$$

referring the D to the functional characteristic f, and the Δ to the o and its powers.—Or, we may throw it into the following form,

$$\left\{\frac{Df(1)}{1}\Delta + \frac{D^2 f(1)}{1.2}\Delta^2 + \ldots\ldots \frac{D^x f(1)}{1.\ldots.x}\Delta^x\right\} o^x$$

Upon this, we have to observe—first, that the addition of the term $f(1)$ at the beginning of the series within the brackets makes no difference in the result; adding only to it the term $f(1)\times o^x$, which vanishes of itself: and, in the next place, that we are at liberty to suppose the series *continued to infinity*; as every term beyond $\frac{D^x f(1)}{1\ldots.x}\Delta^x o^x$ vanishes, owing

to the well-known property of the functions $\Delta^{x+1} o^x$, $\Delta^{x+2} o^x$, &c., each of which is equal to zero. Our series then becomes

$$\left\{ f(1) + \frac{Df(1)}{1} \Delta + \&c. \right\} o^x = f(1+\Delta) o^x$$

and we have therefore

$$f(\varepsilon^t) = f(1) + \frac{t}{1} . f(1+\Delta) o + \frac{t^2}{1.2} f(1+\Delta) o^2 + \&c. \ldots\ldots (2)$$

In applying this series to any particular case we have only to develope $f(1+\Delta)$ in powers of Δ: then striking out the first term, as well as all those where the exponent of Δ is higher than that of t, to apply each of the remaining ones immediately before the annexed power of o, and the developement is then in a form adapted to numerical computation. This formula may be also farther compressed into

$$f(\varepsilon^t) = f(1+\Delta) \varepsilon^{o.t}; \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

by simply writing it as follows:

$$f(\varepsilon^t) = f(1+\Delta) \left\{ 1 + \frac{t.o}{1} + \frac{t^2.o^2}{1.2} + \&c. \right\}.$$

I shall notice one more form in which the same result may be exhibited. If we continue the series (1), as before, to infinity, and add the term 1 at its commencement, it becomes

$$\left\{ 1 + \frac{\Delta . D}{1} + \frac{\Delta^2 . D^2}{1.2} + \&c. \right\} o^x . f(1) = \varepsilon^{\Delta D} o^x . f(1)$$

whence, we obtain

$$f(\varepsilon^t) = f(1) + \frac{t}{1} . \varepsilon^{\Delta . D} o . f(1) + \frac{t^2}{1.2} . \varepsilon^{\Delta D} o^2 . f(1) + \&c.$$

or, attending carefully to the application of the symbols

$$f(\varepsilon^t) = \varepsilon^{\Delta D} \left\{ 1 + \frac{o.t}{1} + \frac{o^2.t^2}{1.2} + \&c. \right\}$$

$$= \varepsilon^{\Delta . D + o.t} f(1); \ldots\ldots\ldots\ldots\ldots (4)$$

We will now proceed to apply these results to the actual

developement of equation (a). And, first, in the case where n is a positive integer, we have

$$f(\varepsilon^t)=(\varepsilon^t-1)^n; \qquad f(t)=(t-1)^n.$$

consequently,

$$f(1+\Delta)=(1+\Delta-1)^n=\Delta^n$$

wherefore the equation (2) becomes

$$(\varepsilon^t-1)^n=\frac{t}{1}.\Delta^n o+\frac{t^2}{1.2}.\Delta^n o^2+\frac{t^3}{1.2.3}\Delta^n o^3+\ \&c.; \ldots\ldots (5).$$

of which the first $n-1$ vanish of themselves.

Let us next consider the formula $(\varepsilon^t-1)^{-n}$; $-n$ being a negative integer. As this function, when developed, must evidently contain the negative powers of t, as far as t^{-n}, we first throw it into the form

$$t^{-n}\left\{\frac{t}{\varepsilon^t-1}\right\}^n, \text{ or its equal } t^{-n}\left\{\frac{\log.\,\varepsilon^t}{\varepsilon^t-1}\right\}^n$$

supposing then $f(\varepsilon^t)=\left\{\frac{\log.\,\varepsilon^t}{\varepsilon^t-1}\right\}^n$, we shall have by applying the equation (2)

$$\left\{\frac{t}{\varepsilon^t-1}\right\}^n=1+\frac{t}{1}.\left\{\frac{\log.\,(1+\Delta)}{\Delta}\right\}^n o+\frac{t^2}{1.2}.\left\{\frac{\log.\,(1+\Delta)}{\Delta}\right\}^n o^2+ \\ \&c.; \ldots\ldots\ldots (6).$$

All that now remains to be done is, to develope the function $\left\{\frac{\log.\,(1+\Delta)}{\Delta}\right\}^n$ in powers of Δ. When $n=1$, the developement is well known to be

$$\frac{1}{1}-\frac{\Delta}{2}+\frac{\Delta^2}{3}-, \&c.$$

Hence, if we suppose

$$\frac{t}{\varepsilon^t-1}=1+B_1.\frac{t}{1}+B_2.\frac{t^2}{1.2}+\ \&c.$$

we shall have

$$B_x=\frac{\log.\,(1+\Delta)}{\Delta}o^x$$
$$=-\left\{\frac{\Delta o^x}{2}-\frac{\Delta^2 o^x}{3}+\ldots\ldots\pm\frac{\Delta^x o^x}{x+1}\right\}; \ldots\ldots\ldots\ldots (7).$$

and in general, if

$$\left\{\frac{t}{\varepsilon^t-1}\right\}^n = 1 + {}^nB_1 \cdot \frac{t}{1} + {}^nB_2 \cdot \frac{t^2}{1.2} + \text{\&c.}$$

we shall have

$${}^nB_x = \left\{\frac{\log.(1+\Delta)}{\Delta}\right\}^n 0^x; \ldots\ldots\ldots\ldots (8).$$

The coefficient of Δ^x in the function $\left\{\frac{\log.(1+\Delta)}{\Delta}\right\}^n$, developed in powers of Δ, is evidently

$$\frac{d^{x+n}.(\log.t)^n}{1.2\ldots\ldots(x+n).dt^{x+n}}$$

t being made $=1$ after the differentiations. Now we easily find, that the expression

$$\frac{d^{x+n}.(\log.t)^n}{dt^{x+n}}$$

After executing the operations indicated must take the form

$$\frac{A_x + {}^1A_x.\log.t + \ldots\ldots {}^{n-1}A_x.(\log.t)^{n-1}}{t^{x+n}}$$

and the equations which determine A_x, &c. are

$${}^{n-1}A_{x+1} = -(x+n).{}^{n-1}A_x$$

$${}^{n-2}A_{x+1} = -(x+n).{}^{n-2}A_x + (n-1).{}^{n-1}A_x$$

$$\cdots\cdots\cdots\cdots\cdots$$

$$A_{x+1} = -(x+n).A_x + 1.{}^1A_x$$

The integration of these equations is attended with no difficulty, and gives for the value of A_x (the only one wanted) as follows:

$$(-1)^x\, 1.2.\ldots.(x+n-1).\Sigma\frac{1}{x+n}\Sigma\frac{1}{x+n}\Sigma\ldots\ldots\Sigma\frac{1}{x+n}$$

where there are $(n-1)$ signs of integration; a constant being included under each. If now we suppose

$$\frac{1}{1} + \frac{1}{2} + \ldots\ldots\frac{1}{x} = S\left\{\frac{1}{x}\right\},$$

$$\frac{1}{1.2} + \frac{1}{1.3} + \frac{1}{2.3} + \ldots\ldots\frac{1}{(x-1).x} = {}^2S\left\{\frac{1}{x}\right\}$$

and so on, we shall have no difficulty in convincing ourselves that

$$A_x = (-1)^x \, 1.2.\ldots.(x+n-1).\,{}^{n-1}S\left\{\frac{1}{x+n-1}\right\}$$

All the constants vanishing but that added at the first integration which is equal to $1.2.\ldots n$. When $t = 1$, the expression for $\frac{d^{x+n}.(\log. t)^n}{dt^{x+n}}$ reduces itself to A_x, and therefore the coefficient of Δ^x will become

$$(-1)^x \frac{1.2\ldots.n}{x+n}.\,{}^{n-1}S\left\{\frac{1}{x+n-1}\right\}$$

We are thus conducted to the following value of ${}^nB_x$,

$${}^nB_x = -1.2\ldots.n.\left\{\frac{\Delta 0^x}{n+1}.\,{}^{n-1}S\left\{\frac{1}{n}\right\} - \frac{\Delta^2 0^x}{n+2}.\,{}^{n-1}S\left\{\frac{1}{n+1}\right\}\right.$$
$$\left. + \ldots\ldots \frac{\Delta^x 0^x}{n+x}.\,{}^{n-1}S\left\{\frac{1}{x+n-1}\right\}\right\} \ldots\ldots\ldots\ldots (9)$$

The cases where $n = 1$ and $n = 2$ are the only ones of sufficient importance to merit a more particular consideration. In the former, we have already in our equation (7) given the expression for ${}^1B_x$ or B_x. Its alternate, even values (the signs alone excepted) are those numbers so well known in analysis by the name of the "Numbers of BERNOUILLI," and among the variety of expressions they admit, I know of none so compendious, or so readily computed arithmetically. Indeed, to compute the higher numbers of BERNOUILLI directly has always been attended with some labour. If we examine the values of B_0, B_1, B_2, &c., we shall observe that all the odd ones (with the exception of $B_1 = -\frac{1}{2}$) vanish: as indeed may easily be shown *a priori* from the nature of the function $\frac{t}{\varepsilon^t - 1}$.

A considerable simplification of the latter case takes place owing to this circumstance: the alternate values of ${}^2B_x$ being

susceptible of an expression by means of those of ${}^{1}B_x$: In fact, the odd values of B_x vanishing (except B_1), we have

$$1 + {}^{2}B_1 \cdot \frac{t}{1} + \&c. = \left\{\frac{t}{\varepsilon^t - 1}\right\}^2$$
$$= \left\{B_1 . t + \left(1 + \frac{t^2}{1.2} B_2 + \&c.\right)\right\}^2$$

and, comparing the coefficients of t^{2x+1} in the two members of this equation, we obtain

$${}^{2}B_{2x+1} = -(2x+1) . B_{2x} .$$

Hence this remarkable theorem,

$$\left\{\frac{\Delta}{3} S\left(\frac{1}{2}\right) - \frac{\Delta^2}{4} . S\left(\frac{1}{3}\right) + \&c.\right\} 0^{2x+1} = -(2x+1) . B_{2x}; \ldots\ldots (10)$$

which may also be regarded as affording another general expression for the numbers of BERNOUILLI.

LAPLACE has shown that the developement of the function $\frac{t}{\varepsilon^t - 1}$ may be derived from that of $\frac{1}{\varepsilon^t + 1}$, and that, if the coefficient of t^x in the developement of the latter be represented by a_x, it will be $-\frac{a_{x-}}{2^x - 1}$ in that of the former. Now, by the application of our equation (2), we find that

$$\left\{\frac{1}{\varepsilon^t + 1}\right\}^n = \left(\frac{1}{2}\right)^n + \frac{t}{1}\left\{\frac{1}{2+\Delta}\right\}^n 0 + \frac{t^2}{1.2} . \left\{\frac{1}{2+\Delta}\right\}^n 0^2 + \&c.; \ldots\ldots (11)$$

Making then $n = 1$, we find for the value of a_{x-1}

$$a_{x-1} = \frac{-\left\{2^{x-2} \Delta - 2^{x-3} \Delta^2 + \ldots\ldots \pm \Delta^{x-1}\right\} 0^{x-1}}{1.2. \ldots\ldots (x-1) . 2^x}$$

and consequently the coefficient of t^x in $\frac{t}{\varepsilon^t - 1}$ will be

$$\frac{\left\{2^{x-2} \Delta - 2^{x-3} \Delta^2 + \ldots\ldots \pm \Delta^{x-1}\right\} 0^{x-1}}{1.2 \ldots\ldots (x-1) . 2^x . (2^x - 1)}; \ldots\ldots\ldots\ldots (12)$$

Dr. BRINKLEY has arrived at the same result.

The computation of the functions ${}^{n-1}S\left\{\frac{1}{n}\right\}, {}^{n-1}S\left\{\frac{1}{n+1}\right\}$ &c. is attended with very little difficulty; for, if we multiply together successively the terms $1 + z$, $2 + z$, $3 + z$, &c. and call

the co-efficient of z^{x-p} in the product ${}^{p}S\,(x)$, we shall have

$$\frac{{}^{x-n+1}S(x)}{1.2.3\ldots\ldots x} = {}^{n-1}S\left\{\frac{1}{x}\right\}$$

and, as every value of ${}^{n-1}S\left\{\frac{1}{x}\right\}$, from $x=n$ up to $x=\infty$ is wanted, the principal part of the work consists in calculating the first n terms of the successive products, which, (being derived from one another) except n is considerable, is attended with very little trouble.

The remarkable form of our equation (2) enables us to exhibit a variety of properties of the functions comprehended under the expression $\Delta^n\, o^x$, some of the principal of which I shall now proceed to notice.

Suppose $f(\varepsilon^t) = a_0 + a_1\,.\,t + a_2\ t^2 + \&c.$
Then, as we have shown,

$$a_x = \frac{f(1+\Delta)\,o^x}{1.2\ldots\ldots x}; \ldots\ldots\ldots\ldots\ldots\ldots \qquad (13)$$

from which we find

$$f(1+\Delta)\,o^x = 1.2.\ldots.\,x.\,a_x.\ldots\ldots\ldots\ldots \qquad (14)$$

If then the developement of $f(\varepsilon^t)$ be given, we are enabled to assign the value of $f(1+\Delta)o^x$ in functions of x, and the converse. It is scarcely necessary, however, to remark, that the extent of these equations is not limited to cases in which the actual developement of $f(1+\Delta)$ in powers of Δ is *practicable*, or in which the form of f is known, or even dependent on analytical relations.

Let us suppose a function $\mathrm{F}(t)$, and any two others $f(t)$ and $f'(t)$, so related that

$$\mathrm{F}(t) = f(t)\,.\,f'(t)$$

Let also

$$\mathrm{F}(t) = \mathrm{A}_0 + \mathrm{A}_1\,.\,t + \mathrm{A}_2\,t^2 + \&c.$$

a similar notation being used, for $f(t)$ and $f'(t)$, changing only A into a and a'. It is evident then, that

$$A_x = a_0 \cdot a'_x + a_1 \cdot a'_{x-1} + \ldots\ldots a_x \cdot a'_0 .$$

In this equation, substituting for A_x &c., their values drawn from (13), we find

$$F(1+\Delta)o^x = f(1+\Delta)o^0 . f'(1+\Delta)o^x + \frac{x}{1} . f(1+\Delta)o^1 . f'(1+\Delta)o^{x-1}$$
$$+ \frac{x(x-1)}{1.2} . f(1+\Delta)o^2 . f'(1+\Delta)o^{x-2}$$
$$+ \text{\&c.}$$

This equation may be abbreviated, upon the principles we have all along adopted, by a very simple and convenient artifice of notation, viz. by applying an accent to one of the Δ and also to the corresponding o; *these* accents not altering the meaning of the symbols, but solely pointing out those which are to be applied to one another. The second number of this equation then becomes

$$f(1+\Delta)o^0 . f'(1+\Delta')o'^x + \frac{x}{1} . f(1+\Delta)o . f'(1+\Delta')o'^{x-1} + \text{\&c.}$$

in which the symbols of operation may now, without confusion, be separated from those of quantity, when it will take the form

$$f(1+\Delta) . f'(1+\Delta') \left\{ o'^x + \frac{x}{1} . o . o'^{x-1} + \text{\&c.} \right\}$$

And our equation becomes

$$F(1+\Delta)o^x = f(1+\Delta) . f'(1+\Delta') \left\{ o + o' \right\}^x ; \ldots\ldots\ldots\ldots\ldots \quad (15)$$

We must here notice, that the second member of this equation is precisely what the first would become, if, instead of $F(1+\Delta)$ we had written $f(1+\Delta) . f'(1+\Delta)$, its *equivalent*, and instead of o the symbolic expression $o+o$ which is *equal* to it in quantity, and then applied the former Δ to the former o, and the latter to the latter, by the method of accentuation

above explained. Pursuing this idea, let us suppose $F(t)$ to be decomposable into any number of factors $f(t), f'(t), f''(t)$, &c., and by executing the same mechanical process on the expression $F(1+\Delta)o^x$, we resolve it into

$$f(1+\Delta).f'(1+\Delta').\text{\&c.}\left\{o+o'+o''+\text{\&c.}\right\}^x.$$

A moment's attention to the method by which (15) was originally derived, will convince us that (attending to the proper application of the symbols) we are at liberty to *develope* the expression $\left\{o+o'+o''+\text{\&c.}\right\}^x$, and thus we have the *equation*

$$F(1+\Delta)o^x = f(1+\Delta).f'(1+\Delta').\text{\&c.}\left\{o+o'+\text{\&c.}\right\}^x \text{.............} (16)$$

Should any one of the functions $f(1+\Delta)$, &c., be of the form $(1+\Delta)^k$ any term multiplied by o^i in the developement of $\left\{o+o'+\text{\&c.}\right\}^x$ will acquire the coefficient $(1+\Delta)^k o^i$, which, being, by (14), the coefficient of t^i in the developement of $(1+\varepsilon^t-1)^k$, or ε^{kt}, multiplied into $1.2.3.\ldots.i$, is evidently equal to k^i. Now it is the same thing whether we write k^i for $(1+\Delta)^k o^i$ *after* the developement, or at once strike out $(1+\Delta)^k$, and for o write k previously to it. Hence we conclude that

$$(1+\Delta)^k.F(1+\Delta)o^x = f(1+\Delta).f'(1+\Delta').\text{\&c.}\left\{k+o+o'+\text{\&c.}\right\}^x; \text{......} (17)$$

where, as before, $F(t) = f(t).f'(t).$ &c.

The expression $f(1+\Delta)o^x$ is susceptible of a somewhat varied form, deducible from the identical equation

$$f(\varepsilon^t) = f\left\{\left(\varepsilon^{\frac{1}{n}t}\right)^n\right\}$$

The coefficient of t^x in the second member of this is equal to that of t^x in $f\left\{(\varepsilon^t)^n\right\}$ multiplied by $\frac{1}{n^x}$, that is, by (13), to

$\frac{1}{n^x} \cdot \frac{f\left\{(1+\Delta)^n\right\}^x o}{1.2\,.\,.\,.\,.\,.\,x}$; and thus we obtain

$$f\left\{(1+\Delta)^n\right\}o^x = n^x f(1+\Delta)o^x\,; \;\ldots\ldots\ldots\ldots\ldots \quad (18)$$

From this equation it is easy to derive the two following

$$o = \left\{f(1+\Delta) + f\left(\tfrac{1}{1+\Delta}\right)\right\}o^{2x-1}\,; \;\ldots\ldots\ldots\ldots\ldots \quad (19)$$

$$o = \left\{f(1+\Delta) - f\left(\tfrac{1}{1+\Delta}\right)\right\}o^{2x}\,; \;\ldots\ldots\ldots\ldots \quad (20)$$

Let $f(\varepsilon^t)$ be a rational, integral, finite function of t, and suppose it to contain the powers of t, t^p, t^q, t^r, &c.; it is evident then that we shall have, by (14)

$$f(1+\Delta)o^x = o\,; \;\ldots\ldots\ldots\ldots\ldots\ldots \quad (21)$$

in every case except where x is equal to either of the numbers p, q, r, &c. The following forms of f satisfy this condition

$$f(t) = (\log.\, t)^n$$
$$f(t) = {}^2L(t) + {}^2L\left\{\frac{1}{t}\right\}$$
$$f(t) = {}^nL(1+t) + (-1)^n\, {}^nL\left\{1+\frac{1}{t}\right\}$$
$$f(t) = {}^nC(t) - (-1)^n\, {}^nC\left\{\frac{1}{t}\right\}$$

or, lastly, the sums, powers, or products of any of these forms, any how combined.* The excepted values of x, are—for the first of these forms, $x = n$—for the second, $x = 2$,—and for the third and fourth, $x = n$, or $n-2$, $n-4$, &c. Also from the general theorems delivered by Mr. SPENCE, we find for the value of $f(1+\Delta)o^{n-2x}_{\;\;\;\;1}$ (which comprehends all the excepted cases) in the third and fourth of the above forms respectively ${}^{2x}L(2)$ and ${}^{2x+1}C(1)$.

It may not be uninteresting to descend to a few more particular applications of these general theorems. If we suppose

* Logarithmic transcendents, pages 45, 69.

$f(t)=(\log. t)^n$, n being a positive integer, we have $f(\varepsilon^t)=t^n$ and consequently, by equation (14),

$$\{\log. (1+\Delta)\}^n o^x = o; \ldots\ldots (22).$$

in every case but where $x=n$, when it becomes $1.2\ldots n$. If $n=1$, this becomes

$$o = \frac{\Delta o^x}{1} - \frac{\Delta^2 o^x}{2} + \ldots\ldots \pm \frac{\Delta^x o^x}{x} \ldots\ldots (23).$$

in every case but where $x=1$

If we take $f(t)=\frac{1}{t}$, or $f(\varepsilon^t)=\varepsilon^{-t}$, we find in the same way

$$1=\Delta^x o^x - \Delta^{x-1} o^x + \ldots\ldots \pm \Delta o^x \ldots\ldots (24).$$

Again, let $f(t) = \frac{2t}{1+t^2}$, then will $f(\varepsilon^t) = \sec\left(\frac{t}{\sqrt{-1}}\right)$, and as the coefficient of θ^{2x} in sec. θ is (as EULER has shown)*

$$\frac{2^{2x+2}}{\pi^{2x+1}} \cdot {}^{2x+1}C(1),$$

that of t^{2x} in sec. $\frac{t}{\sqrt{-1}}$ will be

$$\frac{(-1)^x . 2^{2x+2}}{\pi^{2x+1}} \cdot {}^{2x+1}C(1).$$

which, compared with the expression $\frac{f(1+\Delta)o^{2x}}{1.2\ldots 2x}$, gives

$${}^{2x+1}C(1) = (-1)^x . \frac{\left\{\frac{\pi}{2}\right\}^{2x+1}}{1.2\ldots\ldots(2x)} \cdot \frac{1+\Delta}{1+(1+\Delta)^2} o^{2x}; \ldots\ldots (25).$$

which seems the most compendious form in which this complicated function is capable of being exhibited in finite terms, as well as the most easy of computation in any insulated case.

If $f(t) = \frac{1}{\sqrt{-1}} \cdot \frac{t^2-1}{t^2+1}$, we have $f(\varepsilon^t) = \tan \frac{t}{\sqrt{-1}}$, and

$$a_{2x-1} = \frac{1}{1.2\ldots\ldots(2x-1)} \cdot \frac{2}{1+(1+\Delta)^2} o^{2x-1}$$

* Calc. differentialis. ${}^{2x+1}C(1)$ is used to denote the series

$$\frac{1}{1^{2x+1}} - \frac{1}{3^{2x+1}} + \frac{1}{5^{2x+1}} - \&c.$$

But the coefficient of t^{2x-1} in tan. $\frac{t}{\sqrt{-1}}$ is

$$(-1)^x \frac{2^{2x}.(2^{2x}-1).}{1.2 \ldots\ldots (2x)} B_{2x-1}$$

where B_{2x-1} *here* denotes the x^{th} in order of the numbers of BERNOUILLI. Equating these two values, we find

$$B_{2x-1} = \frac{(-1)^x . 2x}{2^{2x-1}.(2^{2x}-1)} . \frac{1}{1+(1+\Delta)^2} 0^{2x-1}; \ldots\ldots\ldots (26).$$

We will now proceed to consider the developement of any function of the form

$$u = f(\varepsilon^t, \varepsilon^{t'}, \varepsilon^{t''}, \&c.)$$

t, t', t'', &c. being any number of independent variables. The coefficient of $t^x . t'^y . t''^z$. &c. being denoted by $A_{x,y,z,\&c.}$, we have

$$A_{x,y,z,\ldots\ldots} = \frac{d^{x+y+\&c.} u}{1.2\ldots\ldots x \times 1\ldots\ldots y \times \&c. \times dt^x . dt'^y . \&c.}$$

Now, regarding u as a function of ε^t, we have

$$\frac{d^x u}{dt^x} = f(1+\Delta, \varepsilon^{t'}, \varepsilon^{t''}, \&c.)\, 0^x$$

Again, considering this as a function of $\varepsilon^{t'}$, we obtain

$$\frac{d^{x+y} u}{dt^x . dt'^y} = f(1+\Delta, 1+\Delta', \varepsilon^{t''}, \&c.)\, 0^x . 0'^y.$$

(the accents over the Δ, and o, indicating, as before, the application of the symbols)—and so on. Thus we find

$$\frac{d^{x+y+z+\&c.} u}{dt^x . dt'^y . dt''^z . \&c.} = f(1+\Delta, 1+\Delta', \&c.)\, 0^x . 0'^y . 0''^z . \&c.$$

and of course,

$$A_{x,y,z,\&c.} = \frac{f(1+\Delta, 1+\Delta', 1+\Delta'', \&c.)\, 0^x . 0'^y . 0''^z . \&c.}{1\ldots\ldots x \times 1 \ldots\ldots y \times 1 \ldots\ldots z \times, \&c.} \ldots\ldots (27.)$$

LAPLACE has shown,* that, in any function $u_{x,y,z,\&c.}$ of x, y, z, &c. if x be made to vary by α, y by β, &c. simultaneously, the following equation, analogous to (a) will hold good:

* Theorie Analytique des Probabilités, p. 70.

$$\Delta^n u_{x,\,y,\,z,\,\&c.} = \left\{ \varepsilon^{\alpha.\frac{d}{dx} + \beta.\frac{d}{dy} + \&c.} - 1 \right\}^n u_{x,\,y,\,z,\,\&c.} \ldots\ldots (d)$$

Hence the function to be developed is

$$\left\{ \varepsilon.\, {}^{t}\varepsilon^{t'}.\, \varepsilon^{t''}.\ \&c. - 1 \right\}^n$$

n being a positive or negative integer

In the former case, the coefficient of $t^x.\ t'^y.$ &c. is

$$\frac{\left\{ (1+\Delta).\ (1+\Delta').\ \&c. - 1 \right\}^n.\ o^x.\ o'^y.\ \&c.}{1.\,2.\ldots x \times 1.\,2.\ldots y \times \&c.}$$

that is, developing the numerator

$$\frac{\left\{ (1+\Delta)^n.\ (1+\Delta')^n.\ \&c. - \frac{n}{1}(1+\Delta)^{n-1}\ \&c. + \&c. \right\} o^x.\ o'^y.\ \&c.}{1.\,2\ldots\ldots x \times 1.\,2\ldots y \times \&c.}$$

Now, $(1+\Delta)^n\, o^x = n^x$, $(1+\Delta')^n\, o'^y = n^y$, &c. and thus the numerator of this expression becomes,

$$n^{x+y+\&c.} - \frac{n}{1}(n-1)^{x+y+\&c.} + \&c.$$

$$= \Delta^n\, o^{x+y+\&c.}$$

and the coefficient of $t^x.\ t'^y.$ &c. therefore becomes

$$\frac{\Delta^n\, o^{x+y+\&c.}}{1.\,2\ldots x \times 1.\,2\ldots y \times \&c.} = A_{x,\,y,\,\&c.} \ldots\ldots\ldots (28).$$

In the latter case, where the exponent is negative ($= -n$) the function to be developed is

$$\left\{ t + t' + \&c. \right\}^{-n} \left\{ \frac{t + t' + \&c.}{\varepsilon^{t}.\ \varepsilon^{t'}.\ \&c. - 1} \right\}^n$$

the coefficient of $t^x.\ t'^y.$ &c. in the latter part of this expression, is

$$\frac{\left\{ \frac{\log.\left\{ (1+\Delta).\ (1+\Delta').\ \&c. \right\}}{(1+\Delta)(1+\Delta').\ \&c. - 1} \right\}^n o^x.\ o'^y.\ \&c.}{1.\,2\ldots x \times 1.\,2\ldots, y \times \&c.}; \ldots\ldots (e).$$

Now, let us for an instant suppose the expression

$$\left\{ \frac{\log.\left\{ (1+\Delta)(1+\Delta').\ \&c. \right\}}{(1+\Delta)(1+\Delta').\ \&c. - 1} \right\}^n$$

developed in a series of powers of $(1+\Delta)(1+\Delta')$. &c. continued both ways to infinity, (which is evidently possible) and let $K(1+\Delta)^i. (1+\Delta')^i.$ &c. be any term of the developement. The corresponding term in the above coefficient will be $K(1+\Delta)^i o^x. (1+\Delta)^i. o^y.$ &c.——that is, $K. i^x. i^y.$ &c. or $K. i^{x+y+\&c.}$ But it is plain that the performance of the same operations on $\left\{\frac{\log. (1+\Delta)}{\Delta}\right\}^n o^{x+y+\ \&c.}$ would have led to the same result: and we may therefore conclude that the numerator of (e) is rightly represented by this latter expression, whose value we have already determined (equations 8, and 9). The coefficient therefore of $t^x. t'^y.$ &c. in the developement of $\left\{\frac{t+t'+\&c.}{\varepsilon^t. \varepsilon^{t'}. \&c.-1}\right\}^n$, is

$$A_{x,\, y,\, \&c.} = \frac{{}^{n}B_{x+y+\&c.}}{1\,.\,.\,.\,.\,.\,x \times 1\,.\,.\,.\,.\,y \times \&c.}\ ; \ldots\ldots\ldots\ldots (29.)$$

and the same reasoning may be applied to any function of $\varepsilon^t. \varepsilon^{t'}. \varepsilon^{t''}.$ &c. whatever.

Analogous theorems to those we have deduced respecting functions of one variable may easily be deduced from the value of $A_{x,\, y,\, \&c.}$ given in (27). Thus, since

$$f\left\{\varepsilon^{nt}, \varepsilon^{n't'}\ \&c.\right\} = f\left\{(\varepsilon^t)^n, (\varepsilon^{t'})^{n'}, \&c.\right\}$$

we ought to have

$$f\left\{(1+\Delta)^n, (1+\Delta')^{n'}, \&c.\right\} o^x . o'^y. \ \&c. = n^x . n'^y. \ \&c.$$

$$f\left\{1+\Delta, 1+\Delta', \&c.\right\} o^x . o'^y. \ \&c. \ldots\ldots\ldots (30)$$

which, by assigning particular values to n, n', &c. affords an infinite number of theorems analogous to (19) and 20).

Similar theorems respecting the product of two or more functions of ε^t, $\varepsilon^{t'}$, &c. may be derived. For instance, if

$$F(t,t') = F(t,t') \times f_{,}(t,t')$$

we shall have

$$F\left\{(1+\Delta),(1+\Delta')\right\} o^{x}.\, o'^{y} =$$

$$= f\left\{(1+\Delta),(1+\Delta')\right\} \times f'\left\{(1+\Delta_{,}),(1+\Delta_{,}')\right\}(o+o_{,})^{x}.(o'+o_{,}')^{y}, \quad \ldots\ldots\ldots (31).$$

This, as well as other analogous theorems, flows with such facility from the principles above laid down, that it is unnecessary, as it would lead me beyond the limits I proposed, to enter into any detail respecting them.

Let us now consider the developement of a function of the form $f\psi^{n}(t)$, f, and ψ being functional characteristics of a given form, and $\psi^{n}(t)$ denoting the result of $n-1$ successive substitutions of $\psi(t)$ for t in the expression of $\psi(t)$. Let us then suppose, for brevity's sake, $\psi(\log.\ (1+t)) = \varphi(t)$, and equation (3) will give

$$f\psi(t) = f\varphi(\Delta)\,\varepsilon^{o.t} \ldots\ldots\ldots\ldots (f)$$

and for f writing $f\psi^{n-1}$

$$f\psi^{n}(t) = f\psi^{n-1}\left\{\varphi(\Delta)\right\}\varepsilon^{o.t}$$

again for f writing $f\psi^{n-2}$ and for t, $\varphi(\Delta)$ we get

$$f\psi^{n-1}\left\{\varphi(\Delta)\right\} = f\psi^{n-2}\left\{\varphi(\Delta')\right\}\varepsilon^{o'.\varphi(\Delta)}$$

and so on to

$$f\psi\left\{\varphi(\Delta^{(n-2)})\right\} = f\varphi(\Delta^{(n-1)})\,\varepsilon^{o^{(n-1)}.\,\varphi(\Delta^{(n-2)})}$$

Collecting, now, the whole result, and, for the sake of convenience, inverting the order of the accents, we obtain,

$$f\psi^{n}(t) = f\varphi(\Delta)\varepsilon^{o.\varphi(\Delta')+o'.\varphi(\Delta'')+\ldots\ldots o^{(n-1)}.t}\,; \ldots\ldots\ldots\ldots (32)$$

The second number of this equation, actually developed becomes

$$S\left\{\frac{f\varphi(\Delta)o^{\alpha}}{1.2\ldots\ldots\alpha} \times \frac{\{\varphi(\Delta)\}^{\alpha}o^{\beta}}{1.2\ldots\beta} \times \ldots\ldots \times \frac{\{\varphi(\Delta)\}^{\mu}o^{\nu}}{1.2\ldots\ldots\nu}.\,t^{\nu}\right\}(g)$$

S denoting that the sum of all the possible values of the expression within the brackets is to be taken; $\alpha, \beta, \ldots \nu$, (whose number is n) varying through all integer values, separately, from 0 to ∞. Now the several factors which compose this expression are respectively, the coefficients of $t^\alpha, t^\beta, \ldots t^\nu$ in the developements of $f\psi(t)$, $(\psi t)^\alpha$, $(\psi t)^\beta$, ... $(\psi t)^\mu$. Let these coefficients be represented by H_α, ${}^\alpha K_\beta$, ${}^\beta K_\gamma$, ... ${}^\mu K_\nu$, and we shall find

$$f\psi^n(t) = S\{H_\alpha \,.\, {}^\alpha K_\beta \,.\, {}^\beta K_\gamma \ldots\ldots {}^\mu K_\nu \,.\, t^\nu\}; \ldots\ldots\ldots (33)$$

If for instance, $\psi(t) = \varepsilon^t = \log.^{-1}(t)$, we have

$$H_\alpha = \frac{f(1+\Delta)0^\alpha}{12\ldots\alpha},\ {}^\alpha K_\beta = \frac{\alpha^\beta}{1.2\ldots\beta},\ \&c.$$

whence we obtain

$$f\log.^{-n}(t) = S\left\{\frac{f(1+\Delta)0^\alpha \times \alpha^\beta \beta^\gamma \ldots\ldots \mu^\nu}{1\ldots\alpha \times 1\ldots\beta \times \ldots 1\ldots\nu} t^\nu\right\}; \ldots\ldots (34)$$

To take another example, let us suppose the developement of $f\psi^n(t)$ were required, where $\psi(t) = \varepsilon^t - 1$. In this case equation (f) becomes simply

$$f\psi(t) = f(\Delta)\varepsilon^{0.t}$$

and the formula in (32) gives

$$f\psi^n(t) = f(\Delta)\varepsilon^{0\,.\,\Delta' + 0'\,.\,\Delta'' + \ldots\ldots 0^{(n-1)}\,.\,t}$$

In this case also (33) becomes

$$f\psi^n(t) = S\left\{\frac{D^\alpha f(0)\,.\,\Delta^\alpha 0^\beta\,.\,\Delta^\beta 0^\gamma \ldots\ldots \Delta^\mu 0^\nu}{1\ldots\alpha \times 1\ldots\beta \times \ldots\ldots \times 1\ldots\nu}\,.\,t^\nu\right\}; \ldots\ldots (35)$$

which gives, if $f(t) = t$,

$$\psi^n(t) = S\left\{\frac{\Delta 0^\beta \times \Delta^\beta 0^\gamma \ldots\ldots \Delta^\mu 0^\nu}{1\ldots\beta \times \ldots\ldots 1\ldots\nu}.t^\nu\right\}$$

Now $\Delta 0^\beta = 1$, and if, for the sake of symmetry we write $\alpha, \beta, \ldots\ldots \mu$ instead of $\beta, \gamma, \ldots\ldots \nu$, we shall have

$$\psi^n(t) = \mathrm{S}\left\{ \frac{\Delta^\alpha o^\beta \times \Delta^\beta o^\gamma \times \ldots . \Delta^\lambda o^\mu}{1 \ldots . . \alpha \times 1 \ldots . \beta \times \ldots . \times 1 \ldots \mu} t^\mu \right\}; \ldots \ldots \ldots (36)$$

the number of the indices $\alpha, \beta, \ldots . . \mu$, being $n-1$.

It seems hardly necessary, after what has been said, to notice that the developement of any function, such as

$$f\left\{\psi^n(t), {}'\psi^{n'}(t'), \&c.\right\}$$

in which t, t', &c. denote any number of independent variables, ψ, $'\psi$, &c., any functional characteristics, and n, n', &c., any indices, may be accomplished by the same means—or, more conveniently, derived from (33) in the same manner as the formula (27) was obtained from our equation (2); and the result included in a brief and simple expression. The cases however are few, where the results afforded appear, if I may so express it, in their natural form, and it would therefore be useless at present to extend our views farther in this direction.

JOHN F. W. HERSCHEL.

Cambridge, Nov. 17, 1815.

IV. *On new properties of heat, as exhibited in its propagation along plates of glass. By* David Brewster, *LL. D. F. R. S. Lond. and Edin. In a Letter addressed to the Right Hon. Sir* Joseph Banks, *Bart. G. C. B. P. R. S.*

Read January 11, 1816.

Dear Sir,

In two papers published in the Transactions of the Royal Society,* I have given some account of the action of heat in enabling glass to arrange a beam of light, into two oppositely polarised pencils, and I have shown that unannealed glass, in the form of Prince Rupert's drops, possesses distinct optical axes, and acts upon light like all regularly crystallized bodies.

My attention was sometime ago recalled to this subject, in consequence of having discovered that reflection from all the metals, and total reflection from the second surfaces of transparent bodies, produced the same effect as crystallized plates, in separating a beam of polarised light into its complementary tints. I was thus led to believe, that the existence of two oppositely polarised pencils, and the production of the complementary colours, were concomitant effects, and I prepared to examine the truth of this supposition in the case of heated glass. In my early experiments on this subject, I had not observed these colours, as I was not then in the possession of a mode of detecting them, when they formed the lower tints of the

* See Phil. Trans. 1814, p. 436, and 1815, p. 1.

first order of Newton's scale (*Opticks*, B. II. Part II.); but I have since discovered a method of rendering them in every case visible, by their effects in modifying the colour of a standard plate of sulphate of lime.

The results of these experiments, while they confirm the supposition which I had made, have also led to the discovery of many singular phenomena, which constitute a new branch of physics, analogous in its general character to the sciences of magnetism and electricity. The curious properties of light and heat, which are explained in the following paper, and the new views which are unfolded respecting the structure of crystallized bodies, will I trust, attract the notice of the chemist, the mineralogist, and the natural philosopher; while the variety and splendour of the phenomena which it embraces, will recommend it to the attention of those, who value scientific researches merely as subjects of exhibition or amusement.

Sect. I. *On the transient effects exhibited during the propagation of heat along plates of glass, or during its communication from glass to surrounding bodies.*

Proposition I.

When heat is propagated along a plate of glass, its progress is marked by the communication of a crystalline structure, which changes its character with the temperature, and which vanishes when the heat is uniformly diffused over the plate.

If we lay the edge of a plate of glass upon a bar of red hot iron placed horizontally, and transmit through it a ray of light polarised in a plane inclined 45° to the horizon, the light will be depolarised in various degrees in different parts of the glass. When the temperature is made uniform, the glass plate loses its property of depolarisation. In order to prove that an

inequality of temperature is necessary to the developement of this structure, I held a small plate of glass in a pair of hot pincers with globular ends. It instantly acquired the depolarising structure, and lost it when the diffusion of the heat became uniform. I then cooled the glass, and held it a second time in the same pincers, which were now much colder than before: the depolarising structure was again communicated to it as formerly.

The same result was obtained when 12 plates of glass were placed upon a bar of red hot iron.

Proposition II.

When a plate of glass is brought to an uniform temperature considerably above that of the atmosphere, the communication of its heat to the surrounding air, or to other contiguous bodies colder than itself, is marked by the production of a crystalline structure, similar to that which is described under the preceding proposition.

I took three plates of thick mirror glass, and brought them to an uniform temperature by immersion in boiling water. In this state they exercised no action upon polarised light; but when their edges were placed upon a mass of cold iron, the inequality of temperature, occasioned by the abstraction of their heat, produced a crystalline structure at the very edge of the plates, which polarised a bluish white tint of the first order. At a greater distance from the edges, the plates depolarised a lower* tint in Newton's scale. When the plates

* One tint is said to be *higher* than another, when it belongs to a *higher* order, or is at a greater distance from the black, or the commencement of the scale. This explanation is rendered necessary, in consequence of M. Biot's having used this term in the opposite sense.

are held in the air, the same effect is produced, but in a less degree. See Prop. XIV.

Proposition III.

When heat is propagated along a plate of glass, its particles assume such an arrangement that it exhibits distinct neutral and depolarising axes, like all doubly refracting crystals, the neutral axes being parallel and perpendicular to the direction in which the heat is propagated.

When a ray of light polarised in a plane inclined 45° to the horizon, is transmitted through a glass plate DCEF Fig. 1. (Pl. II.) placed upon a piece of hot iron AB, lying horizontally, it is completely depolarised; but when the plane of primitive polarisation is parallel or perpendicular to the horizon, no change is produced upon the polarised ray, an intermediate effect being exhibited in intermediate positions, as in regularly crystallized bodies. Hence DE is the neutral axis, and DF the depolarising axis of the plate.

Proposition IV.

When the depolarising structure is communicated to glass by heat in the manner already described, the glass acquires the property of arranging polarised light into its complementary colours.

The apparatus being arranged as in Prop. III, let the light transmitted through the glass DCEF be analysed by a prism of calcareous spar, or by reflection at the polarising angle from a plate of black glass, having a motion of rotation round the polarised ray. When the plane of reflection from the black glass is perpendicular to the plane of primitive polarisation, the whole surface of the glass plate will be covered with beautiful and highly coloured fringes parallel to CD, as

represented in Fig. 2. (Pl. II.); and when the plane of reflection is moved round 90° from this position, the surface of the glass will be covered with the complementary fringes, the colours gradually passing from the one state into the other during the rotatory motion of the black glass, in the same manner as in crystallized bodies.

The nature and intensity of the tints are represented by the following formulæ, which are the same as those which M. BIOT found for crystallized bodies.*

$$P = O + E \cos.^2 2\,a.$$
$$\Pi = E \sin.^2 2\,a.$$

In these formulæ P represents the ordinary pencil, and Π the extraordinary pencil: O is the coloured tint which preserves its primitive polarisation, and is not acted upon by the crystallized glass: E is the complementary tint which has lost its primitive polarisation by the action of the glass being polarised in an angle equal to $2\,a$: and a is the azimuthal angle which the axis of the plate forms with the plane of primitive polarisation.

PROPOSITION V.

The coloured fringes mentioned in the preceding Proposition, and represented in Fig. 2. consist of six different sets, two exterior, two interior, and two terminal sets. The exterior sets occupy the edges, the interior sets the middle, and the terminal sets the extremities of the glass plate, and each set is separated from its adjacent set by a deep black fringe.

These different sets of fringes are represented in Fig. 2. (Pl. II.) where CDEF is the glass plate, and CD the edge of it which rests upon the hot iron. The first *exterior* or *lateral*

* See BIOT's *Recherches sur la polarisation de la lumiere, p.* 21.

set, is comprehended between CD the edge of the plate and the black fringe MN, and the 2d exterior or lateral set between the opposite edge of the plate FE and another black fringe OP.

The first *interior* or *central* set lies between MN and *a b*, a line equidistant from the two black fringes, and the 2d interior or central set between OP and the same line *a b*. The first exterior set contains a greater number of fringes than the second exterior set, but in the latter they have a greater breadth; and in both these sets the fringes diminish in breadth, as they recede from the black spaces MN, OP. The *terminal* fringes appear at the extremities MO, NP of the plate.* They are separated from the central fringes by a faint black space, which becomes lighter as the tints increase; and from the lateral fringes by a *diagonal* black space bisecting the angles E, C, D, F. As the tints increase in number, the terminal fringes suffer particular changes, which will be described in the second part of this paper.

When the glass plate extends far beyond the heated iron, the terminal fringes are not produced.

Proposition VI.

To explain the successive developement and subsequent extinction of the fringes during the propagation of the heat along the glass plate.

When the plate of glass CDEF, Fig. 2. (Pl. II.) is set upon the hot iron, a fringe or wave of a pale white colour instantly appears along the line CD, and gradually advances upon the

* The terminal fringes are not shown in this figure; but they are represented in Figs. 3, 4, 8, (Pl. II.) 20 and 21. (Pl. III.)

glass, driving as it were before it a dark and undefined wave. Nearly at the same instant a similar but fainter white wave advances from the upper edge EF, driving before it a similar undefined dark wave; and at no perceptible interval of time, another white fringe appears in a very diluted state about the centre *a b*, advancing towards the edges CD and EF. The waves of white light, which have their origin at the edges of the plate, and those which advance to meet them from the middle, have the effect of condensing the undefined dark waves into two black fringes MN, OP. A faint *yellow* wave next appears at CD, encroaching gradually upon the white one, and is followed by *orange* and *red* tints, completing the first order of colours in NEWTON's scale. The colours of the second order next advance in succession, and the same thing happens with all the superior orders, so that *three*, *four*, and sometimes even *nine* or *ten* orders of colours are distinctly seen between MN and CD. When the *green* colour of the second order appears at CD, a wave of *yellow* of the first order is seen at FE advancing upon the plate, and is followed by tints of *orange*, *red*, *purple*, &c. till several orders are distinctly visible between FE and OP; and nearly at the same time another yellow wave developes itself at *a b*, and gradually encroaches on both sides upon the white fringe, but never reaches MN or OP. The *yellow* at *a b* next becomes *orange*, *pink*, *purple*, *blue*, *green*, &c. Each of these colours advances towards MN and OP, but never covers entirely the preceding colour, so that new fringes, sometimes to the number of six or eight, are thus formed between the black spaces.

The *terminal* fringes are developed at the same time, and nearly in a similar manner.

As the heat of the iron becomes more uniformly diffused over the plate of glass, the fringes between MN and CD diminish rapidly in number, and pass off at CD, those which remain always increasing in magnitude. The same effect takes place at EF, but more slowly, so that there is a particular time when the fringes between EF and OP are equally numerous as those between CD and MN. The two interior sets diminish and disappear in a similar manner, the part AB re-exhibiting all its former colours in an inverse order. Nothing is now seen but the white and black fringes, which gradually die away, and at last disappear when the temperature of the glass becomes uniform.

PROPOSITION VII.

The colours of the fringes in all the six sets ascend in NEWTON'S *scale as they recede from the black spaces* MN, OP, *the fringes adjacent to these spaces being composed of the colours of the first order.*

The truth contained in this proposition might have been safely deduced from a comparison of the tints with those in NEWTON'S scale, or with the table of colours which I have found in the rings exhibited by topaz when exposed to a polarised ray.* In order, however, to obtain a more convincing proof, I took a plate of sulphate of lime, which polarised a bright blue of the second order, and combined it with the plate of glass CDEF. When the axis of the sulphate of lime was parallel to the axis CD, the *blue* of the second fringe below MN was converted into *black*, a tint due to the difference of their actions; but when its axis was at right angles to CD, the same

* See *Phil. Trans.* 1814, p. 204.

blue fringe was converted into a *yellowish green,* a tint due to the sum of their actions. Hence it follows, that the blue in the second fringe below MN is a blue of the second order. Similar results were obtained by combining the sulphate of lime with the parts of the glass which produced the other sets of fringes.

Another proof of the proposition was obtained in the following manner. I took two plates of thick glass, and having placed them on a hot iron, as before, I waited till all the fringes had disappeared except the white of the first order. When one of the plates was lifted vertically, so that the portion of the glass CDNM was opposite to *a b*, the two white fringes produced a black tint. When the same plate was depressed till *a b* of the one plate was opposite to CD of the other, the white fringe above CD was also converted into black. This black, however, was not so deep as before, as the white in the exterior fringe is brighter than in the interior one. In the first case, this superiority was compensated by the cooling of the glass at CD, in consequence of its being lifted from the hot iron, whereas in the second case, the cooling had not affected the interior part *a b*. When, on the contrary, the one plate was held in such a position that its fringes were at right angles to those of the other, as shown in Fig. 3, (Pl. II.) the white of the exterior fringes of the one plate combined with the white of the exterior fringes of the other, produced black. The white of the interior fringes of the one plate, when combined with those of the other plate, produced black, and the white of the interior fringes of the one plate, when combined with the white of the exterior fringes of the other, produced a brighter white.

The result of these combinations is the production of a dark cross, as represented in Fig. 3. (Pl. II.) This cross is extremely regular and beautiful when the two plates have the same breadth, polarise the same tints, and have their exterior fringes of the same magnitude at both edges. When some of these circumstances are varied, the cross changes its form in a manner which can easily be ascertained from a previous examination of the separate fringes; but, when the one plate polarises higher tints than the other, the cross is no longer produced. The fringes of the plate which polarises the highest tint, are bent from their rectilineal direction, as represented in Fig. 4. (Pl. II.) As the figures exhibited at the intersection of two plates can always be determined, *a priori*, from a knowledge of the fringes which each plate produces separately, so the nature of the separate fringes, and the rate at which the tints change, may be easily predicted from the figures which are exhibited at the place of intersection. When the tints polarised by the two plates are numerous and brilliant, the intersectional figures are singularly beautiful.

PROPOSITION VIII.

The parts of the plate of glass which exhibit the two exterior sets of fringes, have the same structure as that class of doubly refracting crystals, including sulphate of lime, quartz, &c. in which the extraordinary ray is attracted to the axis, while the parts of the glass, which exhibit the two interior and the terminal sets, have the same structure as the other class of doubly refracting crystals, including calcareous spar, beryl, &c. in which the deviation of the extraordinary ray from the axis, is produced by a repulsive force. The portions between these which produce the black spaces, have an intermediate structure, like those portions of muriate of soda, fluor spar, and the diamond, which are destitute of the property of double refraction.*

In order to establish this singular result, I combined a standard plate of sulphate of lime which polarised a bright blue of the second order, with the different parts of the glass which produced the six sets of fringes. When the axis of the plate of sulphate of lime was parallel to the fringes, or to CD, the blue of the second fringe in the first exterior set below MN, and the blue of the second exterior fringe above OP, were converted into *black*, but when the axis of the sulphate of lime was perpendicular to CD, the blue of the same fringes was converted into *yellowish green*. On the contrary, when the axis of the plate of sulphate of lime was perpendicular to CD, the blue of the second fringe of the first interior set above MN, and the blue of the second fringe of the second interior

* See LAPLACE's valuable Memoir, *Sur la loi de la refraction extraordinaire dans les cristaux diaphanes,* Mem. de L'Institut. 1809.

set below OP, were converted into *black;* but when the axis of the sulphate of lime was parallel to CD, the *blue* of the same fringes was converted into a *yellowish green.* Hence it follows, that the axis of the parts of the glass which form the exterior sets of fringes, is at right angles to the axis of the parts which form the interior sets. The same result is deducible from the second experiment in Prop. VII. Since, therefore, the same effects as those which we have described, are produced by combining crystallized plates taken from the two classes of doubly refracting crystals, as has been ably proved by M. Biot,* we may consider the truths stated in the Proposition as completely established.

Cor. It follows from this Proposition, that a single plate of glass, crystallized by the propagation of heat, and exposed to a polarised ray, exhibits the same variety of phenomena as all the crystals in the mineral kingdom. We have already seen, that it possesses the structure of all the three classes of doubly refracting crystals. But the individual crystals which compose these classes, are distinguished from each other by the magnitude of their polarising forces, and the same variety is exhibited in the polarising forces of the glass, the parts which are adjacent to CD, *ab*, and FE, having the structure which gives the greatest polarising force, and the parts adjacent to OP, MN, the structure which gives the least polarising force.

* See M. Biot's *Mémoire sur la découverte d'une propriété nouvelle dont jouissent les forces polarisantes de certains cristaux.* Mém. de l'Institut, 1814.

Proposition IX.

When the temperature of the source of heat remains the same, the thicknesses of the glass, whether one or more plates are used, which polarise any particular colour, under a perpendicular incidence, are proportional to the thicknesses of thin uncrystallized plates, which would reflect the same colour in the phenomenon of coloured rings.

M. Biot has shown with much ingenuity, that the thicknesses of sulphate of lime, rock crystal, and calcareous spar, which polarise any particular colour, are proportional to the thicknesses of the uncrystallized plates which reflect that colour:* and there was reason to believe that the same law would regulate the phenomena exhibited by heated glass.

I took several plates of glass of various thicknesses, from the thinnest German crown glass, about $\frac{1}{25}$th of an inch thick, to plate glass $\frac{1}{4}$ of an inch thick, and, having placed them all upon a piece of red hot iron, I found that the number of orders of colours which were developed, was nearly related to the thickness of the glass. As these plates, however, had not the same chemical composition, I employed several pieces of thick mirror glass cut out of the same plate. I placed one of these by itself on the hot iron, and marked the particular tint which it polarised in the first order of Newton's scale.

All the rest of the plates having been placed on the hot iron at the same time with the first, I took each of them in succession, and joined it to the first plate; the tints which were thus produced, ascended in the order of colours as the

* See Biot's *Recherches sur la polarisation de la lumière*, p. 53.

number of plates was increased, and were always such as belonged to a thickness taken in proportion to the number in the third column of NEWTON'S scale.

When one plate, for example, polarised at *ab*, a yellow of the first order, two plates gave an *indigo* of the second order, three a *red* of the second order, four a *green* of the third order, five a *bluish red* of the third order, and six a *yellowish green* of the fourth order. Now the numbers representing these tints in NEWTON'S scale, are nearly 4, 8, 12, 16, 20, 24, and the corresponding thicknesses are 1, 2, 3, 4, 5, 6. A variety of other experiments were made with the same result.

PROPOSITION X.

If a number of glass plates of the same form and of the same chemical composition, but of various thicknesses, are placed upon a hot iron, then if two or more of them are combined symmetrically, that is with their edges CD coincident, the colour polarised in any part will be the same as that which would have been polarised by a single plate having a thickness equal to the sum of the thicknesses of the plates; but if the plates are placed transversely, or with their edges CD at right angles to each other, the colour polarised at those parts of the glass, which are similarly situated with regard to the black spaces, is the same as that which would have been polarised by a single plate, whose thickness is equal to the difference of the thicknesses of the two transverse plates or systems of plates.

I took two plates of mirror glass that had different thicknesses, but nearly the same colour, and having cut them into equal rectangular pieces, I found that three of the one had the same thickness as *five* of the other.

These two parcels of plates were then placed upon the hot iron, and the one parcel exhibited the same tints as the other, both in the exterior and interior fringes.

In order to prove the second part of the proposition, I took three parcels, one of *two* plates, another of *four* plates, and a third of *six* plates, all of them having been cut out of the same mirror. I then placed the different parcels upon the hot iron, and when the colours were perfectly developed, I held the system of *four* plates in a position transverse to the system of *six* plates as shown in Fig. 3. (Pl. II.) A broad fringe of blue light of the second order appeared at the intersection of the central lines *a b*, *a′ b′*. The very same colour was polarised by the system of two plates, whose united thickness was equal to the difference of the thickness of the transverse parcels. See Prop. XV.

Proposition XI.

The number and form of the plates of glass remaining the same, the tints which are polarised at the central line a b, and at the edges CD, FE, *Fig.* 2, *(Pl. II.) ascend in* Newton's *scale as the temperature of the source of heat is increased.*

I took a thick plate of mirror glass 6,9 inches long, 2,27 inches high, and 0,163 thick, and having placed it upon a heated iron, which just appeared red hot in the dark, I found that it polarised the *green* of the second order in the *first* exterior set of fringes, and the greater part of the *white* of the first order in the second exterior set of fringes. When the heat was more intense, the same plate polarised the *green* of the third order in the *first* exterior set of fringes.

When 15 plates of mirror glass were placed upon the top

of a tin vessel enclosing water at a temperature of 190°, they polarised a *green* of the second order. The united thickness of these plates was 1.7 of an inch.

When the heat of my hand was communicated to 11 plates of crown glass, they polarised the blue of the first order, and exhibited distinctly the two black spaces. The temperature of the room during these experiments was 64°. Even one plate of crown glass about 0.28 of an inch thick, exhibits the black spaces and the bluish white fringes by the heat of the hand.

The preceding results are neither sufficiently numerous nor accurate to enable me to determine the relation between the thickness corresponding to the highest tint, and the temperature of the source of heat. An apparatus, however, is preparing for me, by which this point will be easily ascertained by obtaining various temperatures from heated oil or mercury. See Sect. II.

Proposition XII.

The number and form of the plates of glass, and the temperature of the source of heat remaining the same, the magnitude of the fringes of the first exterior set depend upon the law of the decrease of temperature in that part of the glass which produces them. The highest order of colours is always developed where the temperature is a maximum, and the tints descend in the scale as the temperature diminishes.

Let CDEF, Fig. 5. (Pl. II.) be a plate of glass, MN one of the black spaces, and the portion CDNM, that which produces the first exterior set of fringes.

The temperatures at the points B, K, G may be represented by the ordinates BD, GH, KL, of the curve TLHD. The

highest tint is polarised at B where the temperature BD is greatest. A lower tint in the scale appears at G, depending on the temperature GH, and a still lower tint at K where the temperature is reduced to KL. As the temperature of the iron RS diminishes, and the diffusion of the heat over the glass becomes more uniform, the temperature at B will be changed to B *d*, the temperature at G to G *h*, and the temperature at K to K *l*. So that the curve will now have the form *m l h d*. When this happens, the fringes grow broader and diminish in number.

When the diffusion of the heat is uniform, the temperatures and consequently the ordinates will every where be equal, and the curve will change into a straight line, in which case, the fringes completely disappear. When the plate CDEF is lifted from the iron, it begins to cool at CD. The fringes pass off at the edge CD, exhibiting a broad fringe of the same tint. The differences of the temperatures now vary less rapidly, and the line TLHD, becomes a curve of contrary flexure, such as TLHVW or TLHX, when the cooling has made greater progress.

I have not been able to ascertain exactly the relation between the thickness corresponding to the polarised tints at different distances from the source of heat, and the temperature of the glass at the same points; but by assuming the most probable law of the decrease of temperature, and comparing it with the magnitude of the fringes, there is reason to believe, that the thicknesses are nearly proportional to the temperature.

The tints polarised at different parts of the glass plate (a section of which is shown in Fig. 6. (Pl. II.) by ACB) will be

represented by the ordinates of some curve *m n o p q*, cutting the axis at the neutral points *n p*. They reach their maximum at *m* and *q*, where they have the same character, and also at *o*, where they have an opposite character; and they vanish at *n*, *p*, the points which correspond to the black spaces.

Proposition XIII.

The upper edge of the plate which polarises the highest tint in the second exterior set of fringes, has received no sensible accession of heat, and the central parts of the plate, which form the two interior sets of fringes, exhibit no variation of temperature connected with the colours which they polarise. When the number and form of the plates of glass, and the temperature of the source of heat remain the same, the magnitude of these three sets of fringes depends upon the law of the decrease of temperature at that part of the glass which produces the first exterior set.

It will be seen from experiments given under a subsequent Proposition, that the depolarising structure is communicated to the upper edge of the plate of glass, even when it is 2, 4, 5, 6, and 7 inches high. In some of these cases, the edge of the glass has the same temperature as the circumambient air, although the heat necessary to produce the same fringe at the lower edge of the plate, is much greater than that of boiling water.

By spreading over the surface of the plate a thin film of oil of mace, which melts with a slight degree of heat, I was enabled to ascertain that there was no particular variation of temperature connected with the tints which were polarised by the three sets of fringes mentioned in the Proposition.

In every case the number of fringes in these sets increased and diminished with the number in the first exterior set. Their breadth also varied with the breadth of the fringes of the first exterior set, and consequently depended on the law of the decrease of temperature in that part of the glass.

Scholium.

The truth contained in the preceding Proposition, will, I have no doubt, be regarded by philosophers, as one of the most extraordinary in physics. The production of a crystalline structure in the part of the glass adjacent to the heated iron, though a curious property of radiant heat, is in no respect hostile to our established notions. But the communication of the same structure to the remote edge of the glass, where there is no sensible heat, and where the corpuscular forces, by which the particles cohere, are not weakened by any approximation to fluidity, and the existence of an opposite structure in the middle of the glass, developing itself on both sides from a central line, are results to which we can find nothing analogous, but in the perplexing phenomena of magnetical and electrical polarity.

Proposition XIV.

When a plate of glass heated uniformly, and having a temperature considerably above that of the atmosphere, receives a crystalline structure in cooling, as described in Prop. II. the parts which produce the four sets of fringes have each a structure opposite to that which they had when the plate was crystallized by the introduction of heat from without. That is, the parts of the glass which afford the two exterior sets of fringes, have the same structure as the class of doubly refracting crystals, in which the extraordinary ray is repelled from the axis, and the parts which form the two interior sets of fringes, have the same structure as the class in which the extraordinary ray is attracted to the axis.

I took 12 plates of mirror glass, and brought them to an uniform heat by laying them successively on their sides and edges upon a bar of hot iron. Having ascertained, by exposing them to a polarised ray, that they had no action upon light, I placed them with their edges upon a cold iron, so as to exhibit distinctly the white fringes of the four different sets. When the axis of a plate of a sulphate of lime, which polarised a blue of the second order, was placed at right angles to the direction of the fringes, the white of the two exterior sets was converted into a *brownish red,* and the white of the two interior sets into a light *green.* The converse of this happened, when the axis of the sulphate of lime was coincident with the direction of the fringes. When the four white fringes are produced by placing the glass upon a hot iron, all these phenomena are reversed, the *green* tint being now produced

instead of the *brownish red,* and the brownish red instead of the *green.*

The same result was obtained by combining glass plates crystallized in these two different ways.

In order to obtain a still more uniform temperature, I took a parcel of 15 crown glass plates, and suspended them in a vessel of boiling water, at some distance from the bottom. As soon as they had acquired the temperature of the water, I lifted them out, and placed them with their edges on a cold iron. The black spaces and fringes immediately appeared, and a yellow tint was visible in the middle of the interior fringes. The interior fringes had the same properties as the exterior fringes described in Prop. VIII. and *vice versa.* This experiment was frequently repeated with the same result.

The fringes produced in this manner, we shall call the *unusual series* of fringes, in opposition to the *usual series,* or those produced by placing cold glass upon a hot iron.

I was now anxious to observe the phenomena that would be presented by inducing the *unusual* series of fringes upon a parcel of plates that already possessed the *usual* series. In order to effect this, I placed the parcel of 15 plates of glass, already mentioned, with their edges on the bottom of a vessel filled with boiling water.

The bottom of the vessel being very hot, communicated to the parcel of plates the *usual series* of fringes, just like a plate of hot iron. When the parcel was taken out and placed upon a cold iron, the *usual* series of fringes was distinctly seen; but after the lapse of some seconds, it gradually disappeared, and was displaced by the *unusual* series advancing from the edges, and occasioned by the cooling of the plates. The

struggle between the advancing and retiring fringes, had a curious appearance. Before the usual series of fringes vanished, the external fringes became broader, while the middle one gradually diminished. The two black spaces met in the middle of the plate, forming a broad undefined dark space, and the new or unusual series were seen advancing from the edges of the glass. At this instant there were two white spaces in the middle of the plate, and two external ones, but one of the middle white spaces quickly died away, and the unusual series was speedily developed.

If plates of glass that exhibit the usual fringes are taken from the hot iron and allowed to cool in the open air, the fringes will gradually pass away as described in Prop. VI. but, as soon as they disappear, or a little before their disappearance, the opposite sets begin to advance upon the plate in the manner already described.

Proposition XV.

When similar fringes of the usual and unusual series are combined symmetrically, the polarised tint is that which is due to the difference of the thicknesses, but when they are combined transversely, the tint is that which is due to the sum of the thicknesses of the plates. When dissimilar fringes of the two series are combined symmetrically, the polarised tint is that which is due to the sum of the thicknesses, but when they are combined transversely, the polarised tint is that which is due to the difference of the thicknesses of the plates that produce them.

The preceding truth was established by combining the fringes produced by 15 plates of crown glass cooled from the heat of boiling water, with those produced by a plate of glass

placed on a hot iron. The effects produced by the transverse combination will be understood from Fig. 7. (Pl. II.) in which C, C, &c. represent the similar fringes where the combined effect is produced, and DD, &c. the dissimilar portions where the difference of the effect is produced. The portions C, C, &c. will therefore polarise tints higher up the scale than any that are polarised by the plates singly, whereas the portion DD, &c. will polarise tints much lower on the scale than any of those polarised by the single plates.

When the tints of the *usual series* are of the same intensity with those of the *unusual series*, the effect of crossing is very beautiful, and is represented in Fig. 8. (Pl. II.) where the portions corresponding to C, C, &c. are obviously affected with high tints in the scale, while those corresponding to DD, &c. are almost entirely black. The tint polarised in the central fringes of the plates, when separate, was the commencement of *yellow*. The combined tint, in the figure of a round circular spot, was a beautiful *indigo*, which appeared at the centre of the square of intersection ABCD. This gradually shaded off through all the lower tints, and terminated in a dark circular fringe. Beyond this fringe, towards the angles A, B, C, D, an opposite set of fringes were seen; but towards the sides nothing but a dark shade was visible. All these phenomena are the necessary results of the principles already laid down.

Scholium.

The phenomena described in the Proposition are the same as those which are exhibited by crossing plates of the two classes of doubly refracting crystals. The former, are however, far more beautiful than the latter.

Proposition XVI.

To explain the effects produced upon the fringes by varying the height of the glass plates.

In order to observe the changes occasioned by increasing the height of the plates, I employed pieces of glass whose height varied from 0.18 of an inch to 8 inches. When the height is very small, and not above 2 inches, the black spaces occupy nearly the position as shown in Fig. 2. (Pl. II.) The fringes are therefore very small, as they must always diminish with the height, but they are remarkably brilliant, and exhibit much beauty in their developement.

When the plates exceed two inches in height, the distances NP, PE, Fig. 2. (Pl. II.) increase much faster than ND; a smaller number of fringes is developed beyond OP, and their brightness is much impaired. (This effect is shown in Fig. 9. Pl. II.) When the plates are 8 inches high, the whole fringe is faintly seen above OP. The colours of the two interior fringes are developed from a line much nearer MN than OP, and the black fringe OP is extremely indistinct. As high plates almost always burst in pieces when the maximum tint is nearly produced, I was obliged to use plates of common window glass, but, on account of its dark green colour, I could not examine the phenomena with much satisfaction; in using parcels of these large plates, much caution is necessary, as there is almost a certainty of some of them bursting with violence during every experiment.

PROPOSITION XVII.

To explain the effect produced upon the fringes by varying the shape of the glass plates.

When the breadth of the plates is very small compared with their height, the black spaces have the form shown in Fig. 10, 11. (Pl. II.) the breadth AB of the former being 0.8 of an inch, and that of the latter 2.25 inches. In Fig. 10. (Pl. II.) the upper black space ABCD was indistinct about CD, and was scarcely separated from the lower black space E 3 F. Coloured fringes appear at 1, 2, 3, and a white space between AB, CD extending pretty high, and growing gradually fainter, the parts 1, 2, have the same tint, and the parts 3, 4, the opposite tint. With a piece of glass whose height AC was $4\frac{1}{2}$ inches, its breadth AB 4 inches, and its thickness $\frac{4}{10}$ of an inch, I obtained an effect similar to what is represented in Fig. 11. (Pl. II.) The lowest part of the black fringe above 2 was $1\frac{1}{2}$ inch above CD, and the space at 4, all the way to the edge AB, was a pale bluish white. There were numerous fringes between C and D. In Fig. 11. (Pl. II.) the black spaces AB CD, and EF were separated by a whitish space 2. The portions 1, 2, 3, had the same tint, and the portions 4, 5, the opposite tint. See Prop. XXXVI.

The form of the black spaces and the fringes varies in general with the outline of the plates. When the lower edge CD, Fig. 12. (Pl. II.) had a waving form, and was placed upon a hot iron, the adjacent black space had likewise a waving form, and the parts M, N, though the most distant from the source of heat, polarised tints higher in the scale than the parts O, P.

In a piece of glass shaped as in Fig. 13. (Pl. II.) when CD was heated, part of the upper black space had the form *a b c*, and the other part *e f*, terminated at *f*.

Proposition XVIII.

To explain the effects produced upon the fringes by an interruption in the continuity of the glass.

If the second exterior and the two interior sets of fringes were caused by the actual communication of heat to the parts of the glass which produce them, there was reason to believe, that they would not be affected by any breach of continuity in the glass, which did not obstruct the progress of the heat. In order to determine this, I broke a plate of glass ABCD, Fig. 14. (Pl. III.) through the middle *m n*, and having obtained a very clean fracture, I placed the upper fragment CD, upon the lower one. This compound plate was set upon the hot iron RS; but no effect was produced on the upper plate, the fringes developing themselves in AB, just as if CB had been removed. When the heat was almost uniformly diffused over AB, CD began to exhibit faint traces of the white fringes, AB now serving as a new source of heat. The very same result was obtained when the two plates were joined by the interposition of *water*, *Canada balsam*, or *rosin*.

I now took a piece of glass ABCD, Fig. 15. (Pl. III.) interrupted by a fissure, or crack *m n* extending a short way into the plate. When the heat was communicated to its lower edge, the fringes were seen above *m n*, as if the crack had not existed, and the depolarised white light appeared condensed at *n*, like a fluid rushing round the point. The crack, however, suddenly extended to *o*; the upper piece of glass

flew off with violence from the lower one, and a black fringe instantly sprung up below the new edge *m o*, just as if the upper part of the glass had never been in contact with the lower part. In another experiment, attended with the same result, the crystalline structure above *m n* instantly vanished when the crack reached *o*, although the two pieces of glass still cohered with some force. When the fissure *m n* was placed vertically, as in Fig. 16. (Pl. III.) the same effect took place as if the two pieces had been separate, and no change was observed by cementing them with Canada balsam.

Instead of fissures, I now substituted deep grooves cut across the glass. A thick plate which had a horizontal groove cut half through it, and extending from edge to edge, was laid upon the hot iron. The white fringes appeared imperfectly above the groove, and an undefined dark wave below it, as if some fluid had been obstructed in its passage through a narrow channel. It is not improbable that this dark wave was occasioned by the combination of two white fringes of different sets. For if BC, DE, Fig. 17. (Pl. III.) be a vertical section of the plate, and AFG the groove, the parts G F *o m* E may be considered as acting like a separate plate, and will therefore have *o p*, *m n* for its black spaces, while the other part, D B C A F *o m*, will also act as a separate plate, and have *t u*, *r s*, for its black spaces. But the white of the exterior fringe of the first of these plates between FG and *o p* will thus be opposite to the white of one of the interior sets in the other plate, and as these are produced by opposite crystallizations, a black tint will be the result of their union. The bursting of the plate in the direction of the groove, prevented any farther examination of the phenomena.

I next took a plate of glass that had a diamond cut across the middle, where the interior white fringes appeared. Having broken it in two, a black fringe instantly arose between the cut *m n*, Fig. 14. (Pl. III.) and the plate displayed all the interior sets of fringes without receiving an additional supply of heat.

I now attempted to bring the two separated surfaces into close contact by grinding them upon each other; but I could not succeed in making them act upon light like a single plate. The following method, however, enabled me to surmount the difficulty, and to obtain some new results.

I took a piece of annealed crown glass of the size represented in Fig. 52. (Pl. V.) about 0.42 inches thick, and 0.5 broad, and having made a notch with a file at the point B, I applied to it a heated iron, which instantly produced a fissure B *b c d*, and intercepted all the incident light by the total reflection which was produced. After standing an hour, this fissure began to disappear, and in the course of a day, it was as completely closed up as if it had never been made. The fissure was frequently reproduced by a hot iron; and it regularly closed, unless when the expansive effect of the heat was capable of separating the surfaces to too great a distance. Sometimes it closed in a few seconds, and at other times a little mechanical pressure was requisite to effect the reunion. When the fissure was open, I laid the glass upon a hot iron, and it quickly produced the fringes shown in Fig. 53. (Pl. V.) where the phenomena are exactly the same as if the two pieces AB, BD, had been completely separated. But when the fissure was closed, and the glass laid upon the hot iron, it exhibited the different sets of fringes as shown in Fig. 54. (Pl. V.) just as if it had been one continuous mass.

Now it is manifest, that the two pieces AB, BD, though they touch one another optically, are not in physical contact, or in the same state in which they were before the fissure was formed. If we were to make several other notches in the glass with a file, it would always break at the place of the fissure; which proves that the force of cohesion has there been weakened, and that the surfaces, though optically in contact, are physically at a distance. The crystallization of the solid AD, as if it were continuous, forms a fine analogy with the curious fact in magnetism, that two bars of steel pressed together at their extremities, may be magnetised as if they had formed only a single bar, and will exhibit a neutral point at the place of junction.

PROPOSITION XIX.

When heat is propagated from the centre of a plate of glass in radial lines, all the fringes and the black spaces form concentric circles, and four black radial spaces, at right angles to each other, diverge from the centre in directions parallel and perpendicular to the plane of primitive polarisation.

I took a large plate of glass, and applied to its centre a ball of red hot iron. The four black radial lines were distinctly seen diverging from each other at right angles, but the two concentric dark spaces were indistinctly developed. I next intended to grind a hole in the centre of the plate, and to place in it a red hot ball, but having discovered a much better method of generating the circular fringes, which will be explained in the next section, I proceeded no farther in the experimental illustration of the Proposition.

If we suppose ABCDEFGH, Fig. 18, (Pl. III.) to be eight

equal plates of glass placed upon the faces of octagonal bars of hot iron, the black spaces and fringes will have likewise an octagonal form, abstracting the effects which take place at the extremities of the plates. Now if light polarised in a plane inclined 45° to the horizon, is transmitted through this system of plates, the fringes will be distinctly seen in the four plates A, C, E, G, because their depolarising axes are all coincident with the plane of primitive polarisation, but no fringes will be seen in the plates B, D, F, H, as their depolarising axes are inclined 45° to the plane of polarisation. If this system of plates be now turned round the centre O, each of them will exhibit its fringes when it comes into the positions A, C, E, G. These fringes will gradually disappear during the motion of the plates into the positions B, D, F, H, where they will cease to be visible.

Let us now suppose, that the hot iron is applied to the centre of a circular plate of glass A B C D E F G H Fig. 19. (Pl. III.) the black spaces will obviously have a circular form *a b c d e f g h* and A B C D E F G H; and as the neutral axis of each elementary plate, into which we may suppose the circle of glass to be divided, is directed to the axis O, the dark positions will still be B, D, F, H, and consequently there will be a black cross B *b f* F, D *d h* H, having its arms inclined 45° to the horizon. This cross will continue in the same position during the rotation of the plate about its centre O, every elementary plate losing its depolarising power when it comes into the lines B *b f* F, D *d h* H.

Proposition XX.

When heat is propagated from two different sources, in contact with the opposite edges of a plate of glass, the different sets of fringes preserve the same character, the only effect of the additional heat being to polarise higher tints in the different sets of fringes.

I placed 12 plates of window glass upon a hot iron, and when the different sets of fringes were distinctly visible, I held another bar of hot iron in contact with their upper edges, and observed higher tints polarised in all the four sets of fringes. Many of the plates, however, burst with great violence, so that I could not perceive the phenomena that took place when the diffusion of the heat became more uniform.

Proposition XXI.

When heat is propagated through calcareous spar, rock crystal, topaz, beryl, the agate and other minerals that have the property of double refraction, no optical change is produced in their structure.

The greatest heat which I could conveniently apply to doubly refracting crystals, produced no change whatever in their action upon light, whether the heat was propagated in the direction of their neutral, or of their depolarising axes. These crystals appear to be in the state of steel bars saturated with magnetism, which cannot acquire any additional impregnation. Being already in a state of perfect crystallization, they are not capable of receiving from heat any addition to their crystalline structure.

PROPOSITION XXII.

When heat is propagated through muriate of soda, fluor spar, obsidian, semi-opal, and other minerals that have not the property of double refraction, they exhibit the same phenomena as heated glass.

A mass of muriate of soda, when laid upon a hot iron, exhibited a yellow of the first order, both in the external and internal fringes. *Fluor spar* was very slightly affected. *Semi-opal* suffered a greater change; and *Obsidian* displayed the fringes as readily as glass. A piece of *Obsidian* of considerable transparency, and about $\frac{1}{7}$ of an inch thick, possessed naturally the fringes produced by heat. It must therefore have been formed by igneous fusion. This specimen, for which I was indebted to Mr. SIVWRIGHT, was cut out of a round mass, and preserved its original outline. It probably was of the first variety discovered by Sir GEORGE MACKENZIE.*

Rosin, gum copal, horn, amber, tortoise shell, the indurated ligament of the *chama gigantea,*† and various other sub-

* Sir GEORGE MACKENZIE has observed, that there are two very distinct varieties of obsidian. One of these transmits light when cut into thin plates, which, however, seldom appear of an uniform degree of transparency. This variety, at a temperature much under that which can be excited in a common fire place by a pair of bellows, swells, and is converted into pumice by the extrication of a gaseous fluid, which Sir GEORGE MACKENZIE and Dr. JOHN DAVY attempted without success to collect. During the experiment, the smell of nitrous acid was very perceptible. The other variety is denser, of the deepest black colour, and is scarcely translucent at the edges of thin fragments. It does not swell on the application of heat, much more intense than what converts the other variety into pumice. Should this note meet the eye of a skilful analyst, he would do a service to mineralogy by examining both varieties, and by comparing the analysis of pumice with that of the pumice formed from the first variety.

† I have been indebted to Dr. FRANCIS BUCHANAN, F.R.S. for this curious substance. It is as hard and transparent, and has as rich a colour as amber.

stances, both of animal and vegetable origin, receive a new structure during the propagation of heat.

SECT. II. *On the permanent effects produced upon glass by the communication of its heat to surrounding bodies.*

The phenomena described in the preceding Section are of the most transitory nature. Every fringe is in a state of perpetual change: one colour quickly succeeds another, and after heat has rapidly developed all the various tints due to its intensity, they repass through the same hues which they exhibited in their formation, and they finally disappear after a slow and gradual decline. In this respect, only, do the phenomena of crystallized glass differ from those of the regularly organised bodies that compose the three kingdoms of nature. The fine display of colours which characterises the action of crystalline laminæ upon polarised light, are in every respect permanent. The same mineral possesses an invariable structure, and patience only is necessary to detect the phenomena which it presents, and to obtain an accurate knowledge of the character and intensity of its action. The coloured fringes of heated glass, on the contrary, are not susceptible of correct mensuration. Where every thing is in a state of change, no fixed character can be seized, and, instead of measuring, it is often difficult to observe their variations. From this perplexity, however, I have been fortunately relieved by the discovery of a method of fixing glass in a crystalline state, and giving it a character as permanent as that of the most perfect minerals. An account of this method, and of the results which it has enabled me to obtain, will form the subject of the present Section.

Proposition XXIII.

When a plate of glass brought to a red heat is cooled in the open air, or is placed with one of its edges upon a bar of cold iron, the different sets of fringes described in Section I. are developed during its cooling, and they have the same character with those which are produced by placing cold glass upon a hot iron. When the cooling is completed, the structure which affords the fringes, becomes permanent, and the colours, when thus fixed, possess the same brilliancy which they displayed during their formation.

When the red hot plate is exposed to a polarised ray, it exhibits at first no action upon light; the tints advance slowly from the edges, and, after the lapse of 12 or 15 minutes, the glass is cooled, and the crystallization complete.

In this way I have formed various plates of glass which possess a permanent structure, and exhibit the phenomena described in the Proposition, but not having obtained a complete series of different heights and thicknesses, I have not yet taken any exact measurements of the fringes.

The following results with four different plates of glass, will convey some idea of the nature of the tints which are developed. All the plates were brought to a red heat so as not to lose their shape, and were cooled by placing their lower edges upon iron of the same temperature as the surrounding air.

	Thickness of the plates.	Maximum tint at the lower edge.	Tint in the middle.	Numbers in Newton's Table corresponding to the maximum tint.
No. 1.	0.1125 inch	Beginning of blue of the 2d. order.	Blue of the 1st. order.	8.7
2.	0.2000	Green of the 3d order.	Beginning of blue of the 2d. order.	16.2
3.	0.2833	Green of the 4th order.	Beginning of purple of the 1st. order	22.7
4.	0.4375	Nearly end of the red of the 5th order.	Pink of the 2d. order.	35.5

By comparing the numbers in the 5th column, which are millionth parts of an inch, with those in the second column, it will be found that the constant factor, by which we must multiply the thickness of any plate of glass, in order to obtain the thickness of the plate which would afford by reflection a tint similar to its maximum tint, is nearly $\frac{1}{12580}$.

It is a curious circumstance, that the permanent fringes have precisely the same character as the transient fringes which are produced by placing glass plates upon a hot iron, while the transient fringes, developed during the cooling of glass plates, have an opposite character.

The limiting temperature at which the former are changed into the latter, is probably that, at which the permanent structure is communicated.

When the glass plates are cooled more at one edge than at another, the fringes are less distinct, and the tints lower at the edge that is least rapidly cooled. This difference becomes more perceptible as the height of the plates is increased.

When the plates of glass are thick, and exposed to a considerable heat, they often lose their polish, and exhibit on their surface a delicate fibrous texture when examined by a microscope. This texture sometimes consists of grooves which exhibit by reflection the coloured images produced by mother of pearl. It also communicates the same property to wax.

Proposition XXIV.

When a plate of glass, crystallized in the manner described in the preceding Proposition, is inclined to the polarised ray in a plane perpendicular to the direction of the fringes, the central tints ascend in the scale of colours, as if the plate had increased in thickness; but, when it is inclined in a plane parallel to the direction of the fringes, the central tint descends in the scale, as if the plate had become thinner. When the plane of inclination forms an angle of 45° with these planes, no change is produced in the tints.

I took a plate of crystallized glass which polarised in the line *a b*, Fig. 2. (Pl. II.) a broad but very faint tinge of yellow; when it was inclined in a plane perpendicular to the direction of the fringes, the tint which it polarised became a dark orange yellow—but, when it was inclined in a plane at right angles to the former, the tint became a pale bluish white. A similar result was obtained, when the colours belonged to higher orders in the scale.

The effect of inclination may be seen more advantageously when two plates that polarise the very same tint, are placed transversely, so as to exhibit the cross represented in Fig. 3. (Pl. II.) By inclining one of the plates, the other is necessarily inclined in an opposite plane, so that the tints of the one *ascend*, while those of the other *descend* in the scale of colours.

The consequence of this is a separation in the middle of the cross, producing two curved black fringes, having the same appearance that is afforded by crossing two plates that polarise different tints.

PROPOSITION XXV.

If a plate of crystallized glass is cut in two pieces by a diamond along the line a b, Fig. 20. (Pl. III.) each of the separate plates will exhibit the properties of a whole crystallized plate. The portion r s o p of the separate plate which had formerly the structure of the attractive class of doubly refracting crystals, has now the structure of the repulsive class; another portion o p which had the attractive structure, has now an intermediate structure, similar to that of muriate of soda, &c. and so on with the other parts of the crystal.*

If a plate of crystallized glass ABCD, Fig. 20. (Pl. III.) is cut with a diamond along the line *a b*, through the central white fringe, the portion *a b* DC has the same structure as the whole plate, as is represented at *r s* GH, Fig. 21. (Pl. III.) a dark space having started up at *o p*, while the other dark space MN has descended to *m n*; the portion *r s p o*, *m n* GH, have now the structure of the repulsive class, and the intermediate portion *o p n m*, that of the attractive class of crystals.

The same change takes place in the upper plate A B *b a*, Fig. 20. (Pl. III.) which has the appearance shown at E F *s r* Fig. 21. (Pl. III.) In one case I found that the fringes in the upper plate were exactly the reverse of those in the under plate.

When the plate is cut perpendicularly to the fringes, an analogous effect is produced. *Terminal* fringes instantly ap-

* See the *Transactions of the Royal Society of Edinburgh*, Vol. VIII. Part I. where the properties of this intermediate class of crystals are described.

pear at the new extremities. A similar, though a more unexpected result, was obtained by breaking in pieces a large plate, in which the crystallization was extremely irregular, polarising here and there a portion of white light. The plate had a small crack in it, and when broken in three pieces, principally along a line nearly parallel to its edge, each piece was regularly crystallized, having the two black spaces with their accompanying fringes of white light.

The same effects are produced when the plate is cut in pieces by a slitting wheel, or has its shape altered by grinding.

The preceding experiments are not easily made, as it is very difficult to cut this kind of glass with a diamond. It generally flies into many pieces as soon as it is scratched, and, when this does not happen, the pieces separate of their own accord, some time after the diamond has been applied.

Scholium.

The truth contained in the preceding Proposition is analogous to the celebrated experiment in magnetism, where the smallest portion detached from the extremity of a magnet, becomes itself a complete magnet, possessing distinct north and south poles. The exhibition of the same phenomena in glass transiently crystallized during the propagation of heat, as described in Prop. XIII., might have been supposed to arise from some new property of heat, which enabled it to act on the remote edge of the glass without any sensible indication of its presence. This opinion, however, is to a certain extent excluded by the results obtained with glass permanently crystallized and having an uniform temperature. Any portion of the glass passes with the utmost facility from one crystalline

structure to the opposite structure, and from one degree of crystallization to another, according to its position with regard to the edge of the plate; and there cannot be an equilibrium among the forces, by which this change is produced, unless the plate exhibits the different sets of fringes which have already been described.

This optical polarity is produced by heat, just as electrical polarity is developed in the tourmaline, and other minerals by the same agent; and there is as much reason to ascribe the production of the optical phenomena to the action of a peculiar fluid, as there is to explain the phenomena of electricity and magnetism by the operation of magnetical and electrical fluids. The optical fluid, as we may call it, may be supposed to reside in all bodies whatever in its natural state, consisting of two fluids in a state of combination, and capable of being decomposed, and fixed in particular parts of a body by the agency of various causes. It would be a waste of time to point out the numerous and striking analogies, which exist between many of the results contained in this Proposition and some of the most interesting phenomena of electricity and magnetism. Some of them will be noticed in the demonstration of a subsequent Proposition.

Proposition XXVI.

When a rectangular plate of glass is brought to a red heat, and cooled as already described, it will acquire such a permanent structure as to exhibit the coloured fringes when polarised light is transmitted through any of the parallel faces by which it is bounded; every rectangular plate being considered as a solid contained by six parallel planes. The depolarising axes are distinctly developed in all these directions, and form angles of 45° *with the common sections of the planes.*

The fringes described in the Proposition are extremely minute, in plates of glass of an ordinary thickness. They consist of the same number of sets, having the same character and properties as those seen through the broad surfaces of the plates, and their maximum tint is generally lower, though sometimes higher, than the maximum tint of the large fringes produced by the broad surfaces. They are in general perfectly regular, even when there is a great degree of irregularity in the form of the large fringes. In a plate of glass which had various breadths, and which polarised a faint yellow of the first order in its central fringes, and a bright blue of the second order in its exterior fringes, the central tints seen through its edges varied with the breadth of the plate, from a faint yellow of the first order, to a deep blue of the second order.

In order to examine with more accuracy the fringes formed by transmitting polarised light through the different faces of a plate of glass, I crystallized a rough parallelopiped of crown glass, which was about three inches long, and half an inch

thick, and when it was properly cut, and polished on a lapidary's wheel, it had the dimensions shown in Figs. 22, 23, and 24, (Pl. III.). The fringes seen through its two broadest surfaces are represented in Fig. 22. (Pl. III.) The maximum tint of the central fringes is the commencement of the *green* of the second order, and that of the exterior fringes a *green* of the third order. In the fringes seen through the edges of the plate, which are shown in Fig. 23. (Pl. III.) the maximum tint of the interior set is a yellow of the second order, and that of the exterior set is a green of the third order. The fringes seen through the ends of the glass plate are very curious, and are represented in Fig. 24. (Pl. III.) where A shows their form when the line AB is inclined 45° to the plane of primitive polarisation, and B their form when the line AB is parallel, or perpendicular to that plane. I have another parallelopiped of flint glass, about 4.3 inches, by 1 broad, and 1 inch deep, which was crystallized when in the form of a cylinder, and afterwards ground into the shape of a parallelopiped. It exhibited the same phenomena as the preceding, and equalled it in the fine display of numerous orders of colours. The beautiful figures produced by crossing these two pieces, surpass in splendour every optical phenomenon that I have seen.

In these and several other specimens of very thick crystallized glass, the maximum tint was always diminished by the operations of grinding and polishing.

The following descriptions of four specimens of crystallized glass will point out the effects which are produced by changing the form of the plate.

No. 1. One of the most curious specimens of crystallized glass which I have obtained, is a parallelopiped about 0.38 of an inch broad and deep, and 1.11 inch long. It depolarises a faint yellow of the first order in the central fringe, when polarised light is transmitted through the faces of the parallelopiped. But when the light is transmitted along the axis of the parallelopiped, and when the lines AC, AB are parallel or perpendicular to the plane of primitive polarisation, the two images formed by calcareous spar exhibit the forms represented in Figs. 25, 26. (Pl. III.) The first of these consists of a black cross surrounded with beautiful fringes of contrary flexure, and has bright green spots of the third order with a little yellow of the same order; their centre at the four angles, A, B, C, D. Figure 25. (Pl. III.) exhibits a form exactly complementary to Fig. 26. (Pl. III.) and remarkable like it for the symmetry of its form. The coloured spots at the angles are now a brilliant pink, with a spot of blue in the middle of them. When the lines AB, AC are inclined 45° to the plane of primitive polarisation, the two images exhibit the forms represented in Figs. 27, and 28, (Pl. IV.)

No. 2. Is another piece of glass of a square form, and 0.3 of an inch thick, it produced the central cross, and exhibited at the angles all the tints up to the blue of the second order arranged in circles, having the blue or maximum tint in the centre. See Figs. 29 and 30. (Pl. IV.)

No. 3. A third plate 0.4 of an inch thick produced the same effect, the angular tints rising in this case to the yellow of the second order.

No. 4. A fourth plate, 1.2 inch thick, produced fringes of contrary flexure like those of No. 1, but rising to the pink of the fourth order.

The terminal and lateral fringes are produced by No. 2, 3, 4, when they are turned round 45°. Their complementary fringes are extremely beautiful.

When No. 2 is combined with No. 3, they produce fringes of contrary flexure like No. 1. The nature and origin of all these fringes are explained in a subsequent Proposition.

PROPOSITION XXVII.

If a rectangular plate of crystallized glass which exhibits the fringes through its edges is inclined to the polarised ray in a plane perpendicular to the direction of the fringes, the central tint will descend in the scale as if the plate had increased in depth; but when it is inclined in a plane parallel to the direction of the fringes, the tint will ascend in the scale as if the plate had diminished in depth.

The result contained in this Proposition was established by the same experiments which are described in Prop. XXIV., the fringes seen through the edges of the plate being used instead of those seen through its broad surfaces. The effects of inclination in these two cases are directly opposite.

PROPOSITION XXVIII.

The regularity in the crystallization of a plate of glass according to one of its dimensions, is not disturbed by any irregularity of its crystallization in another direction.

If a plate of glass is crystallized from a centre, as in Prop. XIX., or if a confused crystallization is induced by cooling it at different places, so that no distinct fringes can be seen when polarised light is transmitted through the broad surfaces of the plate, the fringes seen through its edges will be perfectly

developed, and will possess the same properties as if the whole plate had been regularly crystallized.

Proposition XXIX.

At the extremities A, B of every plate of crystallized glass, there are four portions N, S, N′ S′, at the boundary between the terminal and the lateral fringes, which possess a structure different from the rest of the plate. These portions have their axes inclined to axes of the other parts of the glass. The portions N, N have their axes in the same direction, and S, S′ in a direction opposite to those of N, N′.

When a plate of crystallized glass is exposed to a polarised ray, so that its length in the direction of the lateral and central fringes is parallel or perpendicular to the plane of primitive polarisation, it will exhibit the appearance shown in Fig. 31. (Pl. IV.) where all the lateral, central, and terminal fringes have vanished. Four luminous spots, however, N, S, N′, S′, will be seen at the extremities A, B, exhibiting tints which, in general, vary from the white of the first order to the pink of the second order, and sometimes exceed, and sometimes fall below, the maximum tint of the central fringes. In order to examine the nature of these tints, I took a plate of glass, which when held in the position already mentioned, polarised at the points N, S, N′, S′, a blue of the second order. I then combined with it a plate of sulphate of lime which polarised the same tint, and which had its axis inclined 45° to the plane of primitive polarisation. The resulting tints at the angles N, N′, were black, or that which was due to the difference of their actions, while the resulting tint at S, S′,

was green, or that which was due to the sum of their actions. The same result was obtained when I combined with the above plate the *central* part of another crystallized plate which had the direction of its fringes inclined 45° to the plane of primitive polarisation.

When the axis of the plate of sulphate of lime was turned round 90°, or when the blue tint was taken from the lateral fringes of a plate of crystallized glass, having the direction of its fringes inclined 45° to the plane of primitive polarisation, an opposite effect was produced, that is, the resulting tint of the portions S, S′, was black, and that of the portions N, N′, green.

In two crystallized plates of a square form which afforded the lateral sets of fringes C, D, and the terminal sets A, B, but no central sets, as shown in Fig. 32. (Pl. IV.) the portions N, N′, S, S′, had the structure described in the Proposition. When the plate was held with the line A, B, parallel or perpendicular to the plane of primitive polarisation, it exhibited the phenomenon shown in Fig. 33. (Pl. IV.)

When any plate of crystallized glass, as AB, Fig. 31. (Pl. IV.) is cut through at CD, either by a diamond or upon a lapidary's slitting wheel, new fringes, *n*, *n′*, *s*, *s′* similar to N, N′ S, S′ start up at the new extremities of the plate.

The fringes described in this Proposition may be called the *diagonal fringes.*

Proposition XXX.

In all the phenomena which have hitherto been described, the results are precisely the same, whether the anterior or the posterior face of the glass plate is exposed to the polarised ray; but, in the portions N, N' S, S' the tints change their character, according as one or other of the faces first receives the polarised light.

If the plate *a b*, Fig. 34. (Pl. IV.) has its lower surface exposed to the polarised light, the portions *n*, *n'* exhibit, when combined with sulphate of lime, a tint due to the difference of their action; and *s*, *s'* a tint due to the sum of their action; but when the upper surface is exposed, as in Fig. 35. (Pl. IV.) the portions *s*, *s'* exhibit, in combination with sulphate of lime, a tint due to the difference of their action, and the portions *n*, *n'*, a tint due to their sum. This curious phenomenon arises from the axes of the elementary crystals suffering an angular change of position, amounting to 90°, by turning the other side of the plate to the polarised ray, as shall be more particularly explained in a subsequent Proposition.

PROPOSITION XXXI.

If a crystallized plate a b, Fig. 34. (Pl. IV.) is placed symmetrically above A B, Fig. 31. (Pl. IV.) either with the two anterior or the two posterior faces coincident, or with the anterior face of the one coincident with the posterior face of the other, or with the end a above A or b above B, or with b above A or a above B, in all these positions the tints polarised by the portions N, N′ S, S′ will ascend in the scale of colours, and be that which is due to the sum of the thicknesses of the plates. If the extremity a or b is placed above B or above A, so that the lines AB, a, b, form a continuous straight line, the tint polarised by the combination, will descend in the scale, and be that which is due to the difference of the thicknesses of the plates.

The truth contained in this Proposition, has been established by direct experiment, although it might have been deduced from the Propositions which precede it.

PROPOSITION XXXII.

When the neutral axes of a plate of crystallized glass are parallel or perpendicular to the plane of primitive polarisation, both the exterior and interior sets of fringes vanish, if the polarised ray is incident perpendicularly upon the plate; but, if the plate is inclined to the incident ray, four sets of fringes are developed. They are separated from each other by three black spaces, and the fringes on each side of the central black line have the same character.

When the lateral and the central fringes have vanished, the four diagonal fringes A, B, C, D, Fig. 36. (Pl. IV.) alone

appear at a vertical incidence, but, upon inclining the plate to the incident ray, in the direction of its length OP, three black spaces *mn*, OP, *qr*, are gradually developed. One of them OP passes through the centre of the plate; and between the black spaces are four sets of fringes 1,1; 1,1; 2,2; 2,2; By examining these fringes with a standard plate of sulphate of lime, and with plates of crystallized glass, I found that the fringes 1, 1, 1,1, had the same character as the diagonal fringes A, D, while the fringes 2,2, 2,2, had the same character as the other two diagonal fringes C, B. In one plate, where the maximum tint of the interior fringe was a faint yellow of the first order, the fringes 1, 1, 2, 2, consisted of a *blue* of the first order, and in another plate where the maximum tint of the interior fringe was a faint yellow of the second order, the fringes between *m n* and *q r* consisted of a *green* of the second order.

Proposition XXXIII.

When a plate of crystallized glass is placed on a red hot iron, the number of its fringes is increased. These additional fringes are the same that would have been produced by combining with the crystallized plate an uncrystallized plate of the same form and thickness, and subjected to the same temperature as the crystallized plate. They disappear when the glass cools, but the permanent fringes are not altered unless the heat be very intense, in which case, they suffer a small diminution.

The results described in the Proposition were obtained by placing crystallized plates upon bars of iron of different temperatures. The plate was held out of the heat of the red hot iron, when its effect was combined with that of an uncrystal-

lized plate. The state of the crystallized plate is analogous to that of a bar of steel not saturated with magnetism. It is capable of receiving from heat a much higher degree of crystallization. See Prop. XXI.

PROPOSITION XXXIV.

When a plate of permanently crystallized glass is brought to an uniform temperature in boiling water, or boiling oil, and is then cooled in the open air, the tints descend in the scale, in proportion to the temperature employed, but, they again resume their former intensity when the plate acquires the temperature of the surrounding air.

This diminution of the tints, arises from the production of the transient and *unusual* series of fringes described in Prop. XIV., which, being of an opposite character from the permanent fringes, necessarily causes them to descend in the scale. The effect is here precisely the same, as if the permanently crystallized plate had been combined, when cold, with a hot plate of the same thickness, oppositely and transiently crystallized by cooling.

PROPOSITION XXXV.

When the centre of a plate of glass brought to a red heat is laid upon the summit of a small cylinder of iron standing vertically, it acquires in cooling a permanent structure which exhibits black spaces, and fringes of a circular form, and the black cross exhibited in Fig. 19. (*Pl. III.*)

In a specimen of plate glass crystallized in this manner, the dark spaces and the black cross are very distinctly developed, a yellow tint of the first order appearing between the dark

spaces. When polarised light is reflected from this plate at the polarising angle, the preceding phenomena are very finely displayed. The minute fringes mentioned in Prop. XXVI. are also seen by looking through the edges of the plate, and are not affected by the circular crystallization.

Proposition XXXVI.

When a cylinder of glass is brought to a red heat, and cooled in the open air, it acquires a permanent crystallization, in which the principal sections of all the elementary crystals are directed to the axis of the cylinder.

The phenomena exhibited by transmitting polarised light along a cylinder of this kind, about $2\frac{1}{4}$ inches long, and $\frac{6}{10}$ of an inch in diameter, are shown in Figs. 37 and 38. (Pl. IV.) where A B C D, Fig. 37. (Pl. IV.) is the principal image, and *a b c d*, Fig. 38. (Pl. IV.) the complementary image. The dark cross AC, BD, instead of having its arms inclined 45° to the horizon, as in Fig. 19. (Pl. III.), has them parallel and perpendicular to the horizon, as the light transmitted through the cylinder happened to be polarised in the plane of the horizon. The luminous spaces between the arms of the cross contain about 10 beautiful rings of coloured light. The complementary image *a b c d* is marked with four dark spots, corresponding to the four luminous portions round the central part of the cross, and the outer part has four dark sectors A, B, C, D, corresponding with the light ones in the other image, and formed of small concentric arches of a dark hue, fringed with tints of different colours. In order to see this phenomenon in all its beauty, it is necessary that the polarised ray be exactly parallel to the axis of the cylinder, as the slightest deviation completely destroys the regularity of the figure.

The crystalline structure which exhibits the dark rectangular cross may be imitated, by forming a circle with various sectors of calcareous spar, having the principal sections of each directed to a common axis.

Having had occasion to grind a part of a glass tube into the shape shown in Fig. 39. (Pl. IV.) I was surprised to observe, upon transmitting polarised light along its axis, and analysing it with calcareous spar, that it was depolarised in eight places, 1, 2, 3, 4, 5, 6, 7, 8, Fig. 40. (Pl. IV.) When the line AB was parallel or perpendicular to the plane of primitive polarisation, the tints were of the first order of Newton's scale. The other image formed by the spar, had the appearance shown in Fig. 41. (Pl. IV.) where the dark spots correspond to the white ones in Fig. 40. (Pl. IV.)

In order to discover the origin of these depolarising apertures, I cut another piece out of the same tube and polished the ends of the small cylinder, without grinding off any of the cylindrical circumference. When it was exposed to polarised light, it exhibited the appearance shown in Fig. 42. (Pl. IV.) where A C B D is a dark cross, separating four luminous sectors, and MNOP a dark circular space increasing in darkness towards the points M, N, O, P. If we now suppose the portions C *a b*, D *c d* to be cut off, something like eight luminous apertures will be left, as in Fig. 40. (Pl. IV.) This however is not the cause of the phenomenon. The four apertures on each side of the centre C, are the four diagonal fringes of the square pieces AC, BC, which act as if they were separated at C, the communication being nearly cut off. In this case, the cylindrical crystallization was converted into a rectangular crystallization by changing the shape of the glass. See Prop. XXV.

When polarised light was transmitted through the flat sides of the glass ABCD, Fig. 39. (Pl. IV.) four white spots were depolarised as shown at 1, 2, 3, 4. All these spots have the same bluish white tint, but those marked 1, 2, have their axis at right angles to that of the spots 3, 4.*

The preceding phenomena as explained by the reasoning in Proposition XIX. furnish us with a complete explanation of the appearances exhibited by *oil of mace*, and described in a former paper.† The dark and luminous sectors are obviously produced by circular groups of crystals, having their axes directed to the same centre, and the halo, or nebulous image must be caused by the crystals having a form approaching to that of a sphere. This species of circular grouping is actually seen in a particular kind of *adipocire*, which I have noticed in the Paper already quoted. The axes of the crystals of adipocire, however, are not directed to the same centre, and therefore do not exhibit the same phenomena as oil of mace.

SCHOLIUM.

The results contained in the Proposition, afford the most satisfactory explanation of the optical properties of PRINCE RUPERT's drops described in a former Paper. (See Phil. Trans. 1815, p. 1.) The cleavages which they exhibit in lines converging to the axis of the drop, and in lines concentric with the outer surface, are necessary consequences of the radial crystallization explained in the Proposition, and may be regarded as an ocular demonstration of its truth.

* These spots are the *diagonal fringes* described in Prop. XXIX.

† See *Phil. Trans.* 1815, p. 38, and 49.

Proposition XXXVII.

When a piece of glass is regularly crystallized, every set of lateral fringes which it exhibits is accompanied with another set of an opposite kind, and the forces by which these fringes are produced, are not in equilibrio, unless when two sets of fringes of one character are opposed to two sets of fringes of the opposite character.

The truth of this Proposition is demonstrated by all the preceding experiments. Some apparent exceptions to it will be stated in the Scholium.

Scholium.

The result announced in the Proposition, naturally leads us to point out the striking analogy which subsists between the phenomena of crystallized glass and those of magnetism. In order to avoid circuitous expressions, I shall consider the part of the glass which polarises the highest tint in one set of fringes as a *north pole*, and the part which polarises the highest tint in the opposite set as a *south pole.*

1. When heat is propagated along a plate of glass, or when glass is permanently crystallized by cooling, and exhibits the fringes shown in Fig. 2. (Pl. II.), its poles will be arranged as in Fig. 43. (Pl. IV.) which represents a section of the glass across the fringes. The north poles are situated at N, N′, and there is a south pole in the middle at S′, A, and B being the neutral points corresponding to the black spaces, where the one kind of polarity passes into the other. This arrangement of the poles is precisely the same as that of a

magnetical needle, which has received its polarity by placing the north pole of a magnet upon its centre, and drawing it several times towards the one extremity without returning back again, and afterwards as many times towards the other extremity. The indefinite nature of the poles and fringes, when the plate of glass is high, as described in Prop. XVI. and XVII., and when the heat advances from one edge of the plate, is perfectly analogous to the indefinite polarity communicated to a steel bar, by applying the pole of a magnet to one of its extremities. The same diffused polarity is acquired by hot glass, when one of its edges is cooled much more rapidly than the other. As two distinct poles, therefore, cannot be given to steel, by applying the magnet at one extremity, in like manner a distinct polarity cannot be communicated to glass, either by heating or cooling it solely at one edge, unless when the height of the plate is very small.

Such is the resemblance, indeed, between the two classes of phenomena, that a description of the state and progress of the poles in magnetising a steel bar, is an accurate description of the state and progress of the poles in crystallizing a plate of glass.

2. When a heated plate of glass is cooled in the open air, and produces the transient fringes described in Prop. XIV., the poles are arranged as in Fig. 44. (Pl. IV.) where S, S′ are south poles, and N a north pole in the middle, A and B being the two neutral points. This arrangement of the poles is exactly the reverse of the preceding, and is the same as that which takes place in a needle magnetised in the manner already described, but with the north instead of the south pole.

3. In a plate of glass of the same form and size as Fig. 45.

(Pl. V.) the two preceding structures are combined. It has three black spaces, $m n$, $\mu \nu$, $o p$, the parts D and B have the same structure as that which produces the exterior sets of fringes, and the parts A,C, the same structure as that which produces the interior set in regularly crystallized plates. The poles are therefore arranged in the manner shown in Fig. 46. (Pl. V.) which resembles a magnet with consecutive poles.

4. Out of nearly one hundred pieces of crystallized glass I have found but one which exhibited only two sets of fringes. The piece of glass AB, Fig. 47. (Pl. V.) was intersected in cooling with a crack m E n, which extended completely across the plate. The parts still cohered with such firmness, as not to separate when taken up in the hand. Upon exposing it to a polarised ray, it gave two white fringes E, F, separated by a dark space OP. The two fringes had opposite characters, so that the poles were arranged as in Fig. 48. (Pl. V.) which resembles that of a perfect magnet. This state of the poles, however, is in the case of glass a state of violence, for when the plate broke in two pieces at the crack m E n, the fringes vanished entirely, and it retained no mark whatever of its former crystalline state. The other portion T did not act upon polarised light either before or after the separation. The pressure of the portion T, therefore, had not allowed the other piece of glass to recover from the state of constraint in which it was held.

Proposition XXXVIII.

To explain the origin and form of the different sets of fringes described in the preceding Propositions.

1. *On the fringes produced by rectangular plates.*

It is not easy to ascertain in what manner the various sets of opposite fringes are produced during the heating and cooling of glass, (See Prop. XXXIX.) but it is obvious from the preceding experiments, that when a plate of glass is either transiently or permanently crystallized, all the elementary crystals of which it is composed, turn one of their neutral axes in the direction of the current of heat. The principal axes of the crystals which form the exterior fringes, are parallel to the one edge, and perpendicular to the other. Thus in Fig. 49, (Pl. V.) the axes of the exterior fringes are perpendicular to AD and BC, and the axes of the terminal fringes are perpendicular to AB and DC, while the axes of the interior fringes are parallel to AD and BC.

Let us now consider, what change should take place in the position of the crystals situated at the angles A, B, C, D. An elementary crystal at E will have its neutral axes perpendicular to AD, as it is out of the reach of the forces which act upon the crystals at the edges AB, DC; but, a crystal G in the diagonal AH, BH being similarly situated with respect to the edges AB, AD, will have a tendency to turn its axis both in the direction AB, and in the direction AD, and being unable to obey both these solicitations, it will turn it in the direction of the diagonal AH, forming angles of 45°, with the axes of all the other crystals of which the glass is composed. Any

other crystal *a* situated out of the diagonal AH, will be acted upon by forces proportional to its distances *am*, *an*, from the edges AB, AD, and in the direction of these lines. It will therefore turn its axes in the direction *a* A the diagonal of the parallelogram A*nam*. In like manner it may be shown, that all the other crystals will turn their axes towards A in lines diverging from A as a centre. Each angular portion, therefore, exactly resembles an inverted quadrant of the cylindrical piece of glass represented in Fig. 37, (Pl. IV.) and described in Prop. XXXIV., and consequently an arm of the black cross will appear in the diagonal AH in every quarter of a revolution. The diagonal portions AH will be dark when all the other fringes are visible, and the diagonal fringes will appear in their full beauty, when the rest have vanished. Since the diagonal fringes at A and C have their axes AH, CH parallel, they will exhibit tints of the same character, and opposite to those of B and D which have their axes BH, DH at right angles to the former. The reason is therefore manifest, why each diagonal fringe changes its character by inverting the plate, for when this inversion takes place the axis of the diagonal portion is put into a position at right angles to its first position.

These observations enable us to explain the appearances shown in Fig. 10, and 11, (Pl. II.) and described in Prop. XVII. In Fig. 10, where the plate is narrow, the black spaces at C and D, bisecting the angles, interfere and nearly obliterate the interior fringes, but in Fig. 11, where the plate is considerably broader, the influence of the angular crystallization does not extend so far, and therefore the interior fringes are seen at 2, Fig. 11. The state of the crystallization at the angles

A, B. C, D, Fig. 49, (Pl. V.) is also peculiar. The glass cools more rapidly there than at any other part, and therefore a higher tint is developed at the angles, than towards the middle of the plate.

2. *On the fringes produced by square pieces of glass.*

If the breadth of the glass plate is equal to its length, as in Fig. 32, (Pl. IV.) all the four diagonal portions nearly meet, and therefore, when the lateral and terminal fringes are developed, the central part is altogether black, as the central fringes have entirely disappeared. When the line AB is parallel or perpendicular to the plane of primitive polarisation, the diagonal fringes appear as in Fig. 33, (Pl. IV.) being always separated from each other by a black space in the form of a cross. This black cross is a necessary accompaniment of the diagonal fringes, for it follows, from the reasoning in Sect. I. of this proposition, that all the crystals situated in the central lines, AB, CD, have their neutral axes in the directions AB, CD, and therefore cease to depolarise the incident light when the diagonal fringes are in full perfection.

3. *On the fringes produced by cylindrical pieces of glass.*

As the heat radiates most copiously from the heated cylinder, in lines perpendicular to its surface, that is, in lines directed to its axis, it follows that the axis of all the elementary crystals will be directed to the axis of the cylinder. The uniformity of the radiation in every part of the cylinder, will produce an uniformity of structure, which will develope similar tints at similar distances from the axis, and thus produce fringes concentric with the cylindrical circumference. The effect of a radial crystallization combined with an angular

crystallization is shown in Fig. 50, (Pl. V.) where ABCD is a plate of glass cooled upon a cylinder of iron at its centre. See Fig. 19, (Pl. III.) and 29, (Pl. IV.)

When the section of the glass is a polygon of any number of sides, the form of its fringes may be easily deduced from the principles which have already been established. When the section is a triangle, no regular figure is seen. If the triangle is equilateral, the lines which bisect the angle, and those which are perpendicular to the sides, are inclined to each other 120°. So that the axes of the crystals are not symmetrically related to the rectangular axes of the particles of light. When the glass is a sphere, the axes are all directed to its centre.

PROPOSITION XXXIX.

To ascertain the probable mechanical condition of the parts of the glass that produce the different sets of fringes.

I have not felt myself authorised to deduce, from any of the preceding results, the mechanical condition of the parts of the glass which produce the different sets of colours. It is obvious that in the case of a red hot plate of glass, cooled in the open air, there is a variable density diminishing from all the edges inwards, but in the propagation of heat along a cold plate, there is no direct argument to prove, that such an increasing density exists at any of the edges, excepting the one adjacent to the source of heat. A similarity, however, in the mechanical conditions of the two plates, may be safely inferred from the perfect similarity of their optical properties. The central part of the crystallized plates, which produce fringes of an opposite character, are in a state of dilatation decreasing

from the central line to each of the black fringes.* This inference is not founded on any direct experiment, but it derives a support almost amounting to demonstration, from a series of new experiments, which I shall soon have the honour of submitting to the Royal Society. These experiments were made by altering the mechanical state of parallelopipeds of animal jellies, both by gradual induration, and by the application of variable pressures; and I have in this way obtained results analogous to those which are described in the preceding paper. In every case the compression of the jelly produced a set of fringes of an opposite structure to those which are occasioned by expansion, and every compression was accompanied with a corresponding dilatation. In like manner it will be found, that there is in all crystallized bodies a variation of density related to their axes, and connected with their polarity, which affords an easy explanation of the fringes of different forms which are exhibited by the various crystals of the mineral kingdom.†

* The appearance of the fracture of glass across the fringes, whether it is transiently or permanently crystallized, is very instructive. It has always the same aspect, and plainly indicates the different mechanical states of the different parts of the glass. From this cause crystallized glass is incapable of being cut with a hot iron, like glass of uniform density, and there is only one way in which the division of the plate can be effected.

† Since this Paper was written, I have discovered that glass, and all other substances that have not the property of double refraction, are capable of receiving it from mechanical pressure, and that a compressing force always produces the structure which gives the exterior fringes in crystallized glass, while a dilating force produces the structure which developes the interior fringes. We are, therefore, entitled to conclude that the middle parts are in a state of dilatation, and the external parts in a state of compression. By a peculiar application of the compressing forces, I have even succeeded in obtaining uniform tints like those produced by plates of sulphate of lime of equal thickness.

Proposition XL.

Radiant heat is not susceptible of refraction, and is incapable of permeating glass like the luminous rays.

The propagation of radiant heat along glass can be rendered visible to the eye by the methods described in the first section of this paper. It advances from the heated edge of the plate, crystallizing the glass during its passage, and producing changes in those parts of the plate where it does not exist in a sensible state.

If the radiant heat is received upon a convex lens, the very same effect is produced. Instead of being bent, like light, at the convex surfaces, it advances, whatever be the angle of incidence, in lines perpendicular to that surface, crystallizing the glass in its progress; and, as soon as it has reached the second surface, it is again discharged, as if from a new source of heat. This experiment I conceive to be an ocular demonstration of the first part of the Proposition.

Dr. Herschel, in his celebrated inquiry into the properties of invisible heat, has deduced the very opposite result from several experiments; but, independently of the minuteness of the effects which he observed, it is manifest, that the thermometer placed in the focus of his lens, received its heat by radiation from the lens itself; and it is also demonstrable, that a convex lens, radiating heat at an uniform temperature, will produce a greater effect upon a thermometer placed in its axis, than upon another having a different position. From the form of the lens, the edges are always the coldest, giving out their heat to the metallic ring in which they are placed,

and therefore, the discharge of heat must be most copious in the direction of the axis.*

The inability of radiant heat to pass through glass, may be considered as a consequence of its refusing to yield to the refractive force; for we can scarcely conceive a particle of radiant matter freely permeating a solid body, without suffering some change in its velocity and direction. The ingenious experiments of M. PREVOST of Geneva, and the more recent ones of M. DELAROCHE, have been considered as establishing the permeability of glass to radiant heat. M. PREVOST employed moveable screens of glass, and renewed them continually, in order that the result which he obtained might not be ascribed to the heating of the screen; but such is the rapidity with which heat is propagated through a thin plate of glass, that it is extremely difficult, if not impossible, to observe the state of the thermometer, before it has been affected by the secondary radiation from the screen. The method employed by M. DELAROCHE of observing the difference of effect, when a blackened glass screen, and a transparent one, were made successively to intercept the radiant heat, is liable to an obvious error.

* The circumstance of the glass cooling most rapidly at the edges, which may be proved by exposing it to a polarised ray, enables us to account for the anomalous and hitherto unexplained fact observed by the younger EULER, that the focal length of a lens is shortened when its temperature is increased. The observation having always been made when the lens was actually cooling, the density, and consequently the refractive power had increased towards the circumference of the lens, and therefore its focal length was diminished.

Might not the spherical aberration of lenses be diminished, and even corrected, by giving them a variable density from their vertex? I have three object glasses of this kind, two crystallized and one uncrystallized, and ground carefully upon the same tool; but I have not yet been able to examine their optical properties.

The radiant heat would find a quicker passage through the transparent screen, and therefore, the difference of effect was not due to the transmitted heat, but to the heat radiating from the anterior surface. The truth contained in M. Delaroche's *fifth* Proposition is almost a demonstration of the fallacy of all those that precede it. He found that "a thick plate of glass, though as much, or more permeable to light than a thin glass of worse quality, allowed a much smaller quantity of radiant heat to pass." If he had employed very thick plates of the purest flint glass, or thick masses of fluid that have the power of transmitting light copiously, he would have found that not a single particle of heat was capable of passing directly through transparent media.

Proposition XLI.

To construct a chromatic thermometer for measuring differences of temperature below that of fluid glass, by the optical effects which they produce.

Differences of temperature have hitherto been measured by the expansions and contractions which they produce in solid, fluid, or gaseous bodies, and all the various thermometrical instruments that have been constructed, differ from each other only in the method by which these mechanical effects are rendered visible. The experiments contained in the first Section of this Paper, present us with an entirely new principle for the construction of a thermometer. We have there seen, that the tints polarised by a plate of glass, increase with the temperature by which they are produced, and therefore these tints may be used as a measure of the temperature, after

the tints, corresponding to several points in the thermometrical scale, have been accurately ascertained.

An instrument of this kind which I have constructed, is represented in Fig. 51, (Pl. V.) where ABC is a series of 20 plates of glass, whose length AB is 3.2 inches, their breadth 1.2 inches, and their united thicknesses BC 5.4 inches. A metallic vessel, DEFG, has its bottom formed of a thin layer of tin or lead, or any other suitable metal which can be poured in a fluid state upon the upper edges of the glass plates, so as to touch them in every part. This perfect contact may be obtained for higher temperatures, by grinding the bottom of the metallic vessel till it touches the edges of the glass in every point.

When a heated fluid is poured into the vessel DEFG, its heat will be instantly communicated to the edges of the plates, and when exposed to a polarised ray, subsequently analysed by reflection from a transparent body, they will exhibit the coloured fringes at AB. Now every tint in the scale of colours has a corresponding numerical value, which becomes a correct measure of the temperature of the fluid.

Instead of pouring the fluid into the vessel, we may remove the vessel altogether, and plunge the glass plates into the fluid. They must then be taken quickly out and suspended in a position where they are properly exposed to polarised light. The maximum tint which they develope at the centre, while cooling, is a measure of the temperature which they have acquired in the fluid.

In order to obtain some idea of the nature of the scale, I made the following trials.—The heat of my hand when applied to the edges of 20 plates of glass, produced instantly the fringes with the black spaces. With 12 plates I have

produced the yellow of the first order; and when one plate only was used, the black spaces, and the bluish white fringes were distinctly visible. A temperature of about 80°, that of the glass being 60°, when applied to 20 plates, polarised in the central fringe a *yellow* of the first order, which corresponds to a tint whose value is 4 in the scale of colours. Hence, one plate would have produced a tint corresponding to $\frac{4}{20} =$ 0.20 of the scale.

When one of the plates was placed upon a bar of red hot iron, just visible in daylight, it polarised in the central fringe the commencement of the green of the second order, which corresponds to 9.35 in the scale.

Now the difference of temperature answering to 0.20 was $80° - 60° = 20°$. Hence we have

$$\text{As } 0.20 : 9.35 = 20° : 935°$$

the difference of temperature of the iron and the glass. The temperature of the iron is therefore $935° + 60° = 995°$.

If we suppose the tints to be so indefinitely marked that the eye can only observe units of the scale of colours, we shall, even in this case, have a scale of 187 to measure the temperature of $935° - 20° = 915°$, which is a scale having each of its divisions equal to nearly 4°. 9. The tints, however, are much more definite than we have supposed, for in the second order of colours, in which the observations may always be made, the eight different tints have the following measures.

Tints.	Values.
Violet - - -	7.20
Indigo - - -	8.18
Blue - - -	9.00
Green - - -	9.71
Yellow - -	10.40
Orange - - -	11.11
Bright red - -	11.83
Scarlet - - -	12.67

Now the difference of the values for violet and scarlet is 5.47, corresponding to *seven* different colours. Hence, upon the supposition that the eye can distinguish merely these separate colours, the accuracy of the scale is increased in the ratio of 5.47 to 7, that is, from 187 to 239, which gives 3°.83 for the value of each unit.

It is quite manifest, however, that we can distinguish at least three points in the developement of each colour; and even if this could not be accomplished by the unassisted eye, it can readily be effected, to a much greater extent, by crossing the fringe with a standard crystallized plate, and observing the degree of curvature which is produced in the fringes. This standard plate may be shaped like a wedge, so as to exhibit the variation of its tints to a great degree of minuteness. In a wedge of this kind, two inches long, and ground out of a crystallized parallelopiped, so as to have an angle of 8°, the highest tint is between the *blue* and the *white* of the first order, corresponding to 2.20 of the scale, and the lowest tint is between the *black* and the *blue*, corresponding to about 0.8. We have therefore a scale of nearly 2 inches to measure a variation in the tints amounting to $2.20 - 0.80 = 1.40$. The

method of using the wedge or nonius is shown in Fig. 55, (Pl. V.) where AB is the wedge, exhibiting tints which vary in intensity from A to B. If we wish to ascertain the tints of a piece of crystallized glass CD, it must be held as in the figure, and moved from A to B. When it has the position CD, the intersectional figure is open horizontally, which shows that the tints of AB, at the point *m*, are higher than those of CD. In the position GH the figure is open vertically, and therefore the tints of the wedge at *o* are lower than those of the plate. But in the intermediate position EF, a dark cross is produced, which evinces the perfect equality between the tints of the wedge at *n* and those of the plate EF. In this manner all tints may be compared with each other, and referred to the scale of colours.

By forming wedges of crystallized glass in this way, we are enabled to observe the gradations by which the tints pass into each other, and to perform many experiments on the orders of colours, which would otherwise have been impracticable.

The sensibility of the preceding instrument depends on several other causes. 1st. On the intensity of the polarised pencil. 2d. On the transparency of the glass. And 3d. On the removal of all internal reflections at the junction of the plates. In the instrument with 20 plates already mentioned, the glass has a green tinge, and the polarised light suffers no fewer than 40 reflections before it reaches the eye. In order to remove these evils, the light should be polarised by reflection from several of the thinnest and most colourless plates of glass that can be procured, so that each plate may polarise and reflect the light which is transmitted through the plate

immediately above it. In this way, I have obtained a light as brilliant as that which is reflected from silver. The internal reflections may be removed by interposing a film of oil between each of the plates, so as to rise above that part of the plate where the tint is to be examined.

If the instrument is properly constructed, with these precautions, I have no hesitation in saying, that it will distinctly mark a difference of temperature equal to 1° of FAHRENHEIT'S thermometer.*

I have thus endeavoured to give a brief view of the numerous experiments which have led to the general results unfolded in the preceding enquiry. The length to which this paper has extended, has prevented me from describing many phenomena, and detailing many experiments, which, though interesting in themselves, did not appear absolutely necessary to the establishment of general views.

Had I included in the demonstration of every proposition, the various experimental proofs which I had actually obtained, this Paper would have swelled to a size which would have rendered it unfit for the consideration of the Royal Society; I have, therefore, selected such experiments as appeared most striking, and have left the detail of the rest, and the representation of many of the phenomena, for a separate work which I propose to publish on the subject.†

* This thermometer possesses advantages peculiar to itself, in enabling us to measure the intensity of the heat produced by the friction of any two substances whatever. When glass is one of the substances, the method of employing the instrument is obvious. When any other substance is used, it must be fixed, without cement, to the lower edge of one or more plates of glass, so that its rubbing surface may be as near as possible to the edge of the glass.

† There is one practical result of the preceding experiments, which deserves particular notice. All articles made of glass, whether they are intended for scientific or

I cannot conclude this paper without expressing my obligations to the Rev. Dr. MILNER of Cambridge, for the very handsome manner in which he transmitted to me a quantity of thick plate glass, which I found it impossible to procure from any other quarter. I was thus enabled to obtain several new results, and to complete many experiments that had been left imperfect.*

I have the honour to be, &c.

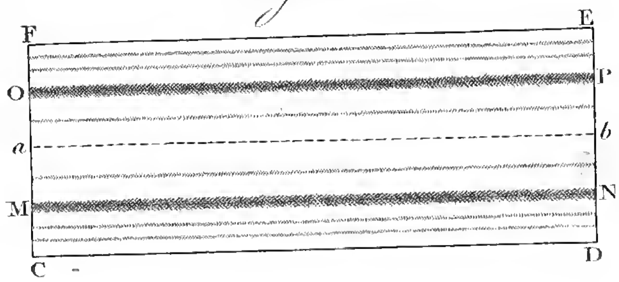
DAVID BREWSTER.

To the Right Hon. Sir Joseph Banks, Bart. G. C. B. P. R. S. &c. &c. &c.

domestic purposes, should be carefully examined by polarised light before they are purchased. Any irregularity in the annealing, or any imperfections analogous to what workmen call *pins* in pieces of steel, will thus be rendered visible to the eye, by their action upon light. The places marked out by these imperfections, are those where the glass almost always breaks when unequally heated, or when exposed to a slight blow. Hence, glass-cutters would find it of advantage to submit the glass to this examination before it undergoes the operations of grinding and polishing.

* Since the preceding letter was written and sent to Sir JOSEPH BANKS, I have learnt that M. SEEBECK has published in a German Journal for Dec. 1814, an account of some experiments similar to those contained in Sect. II. of this Paper. As there is, so far as I know, only one copy of this Journal in England, in the possession of Dr. THOMSON, I have not been able to obtain a sight of it, in order to compare M. SEEBECK's results with mine. I understand, however, that he has discovered the fact, that a plate of red hot glass often acquires, in cooling, the depolarising structure, and that the tints depend upon the mode of cooling the glass. This result, however, has no connection whatever with the new properties of heat unfolded in the first Section of the preceding Paper, and does not anticipate the developement of the phenomena contained in the Second Section. The discovery of the new property of heat was made by me early in 1814, and an account of it was read before the Royal Society on the 19th of May, 1814. See *Phil. Trans.* 1814, p. 436.

Fig. 2.



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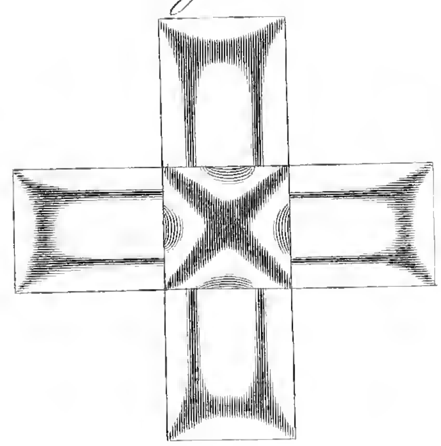

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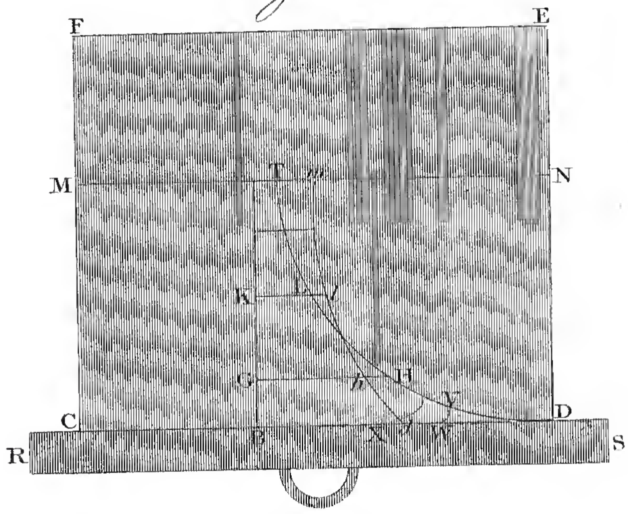

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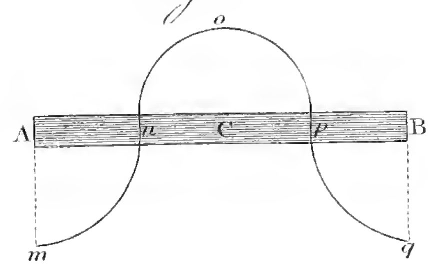

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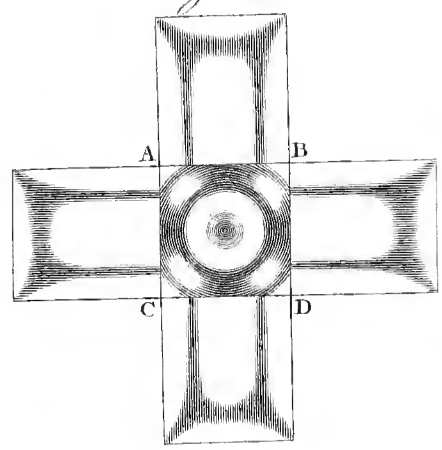

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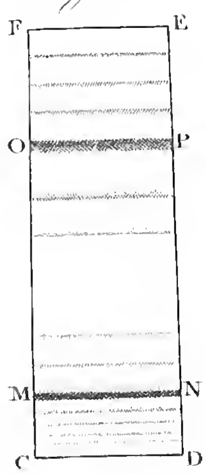

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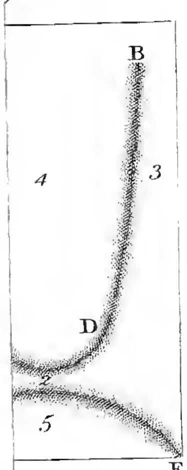

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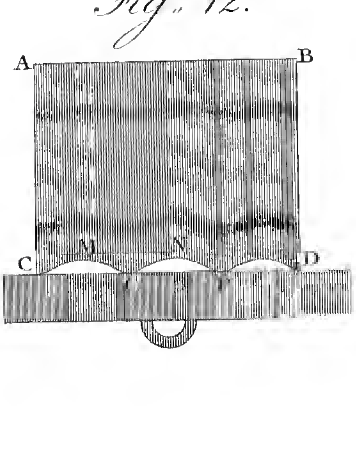

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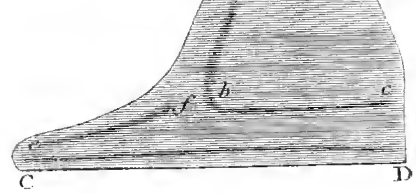

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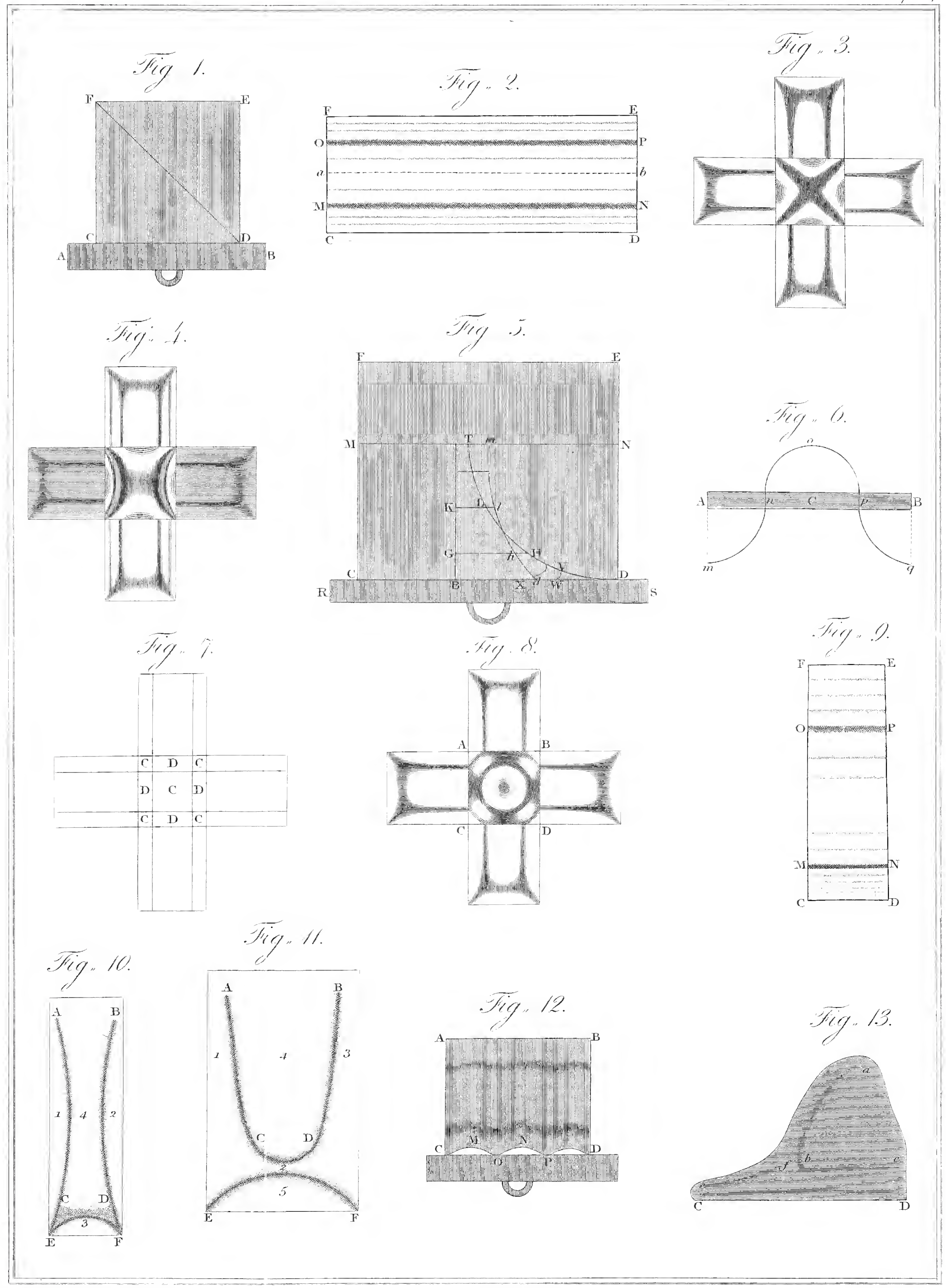
Fig. 1.
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Fig. 2.
F E O P a b M N C D
Fig. 3.
Fig. 4.
Fig. 5.
F E M N T m K L l G H h X Y d W C D B R S
Fig. 6.
o A n C p B m q
Fig. 7.
C D C D C D C D C
Fig. 8.
A B C D
Fig. 9.
F E O P M N C D
Fig. 10.
A B 1 4 2 C D 3 E F
Fig. 11.
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A B M N C O P D
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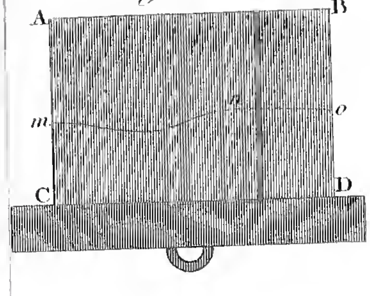
Fig. 15.



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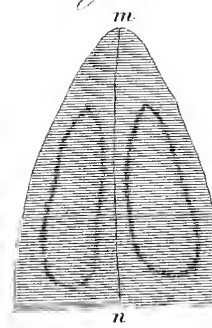

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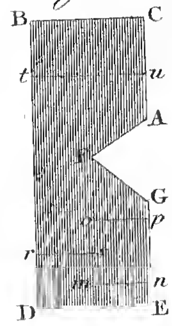

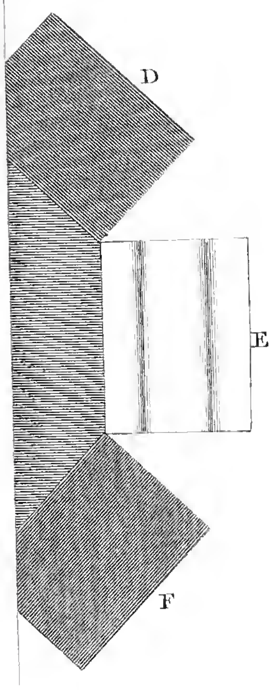

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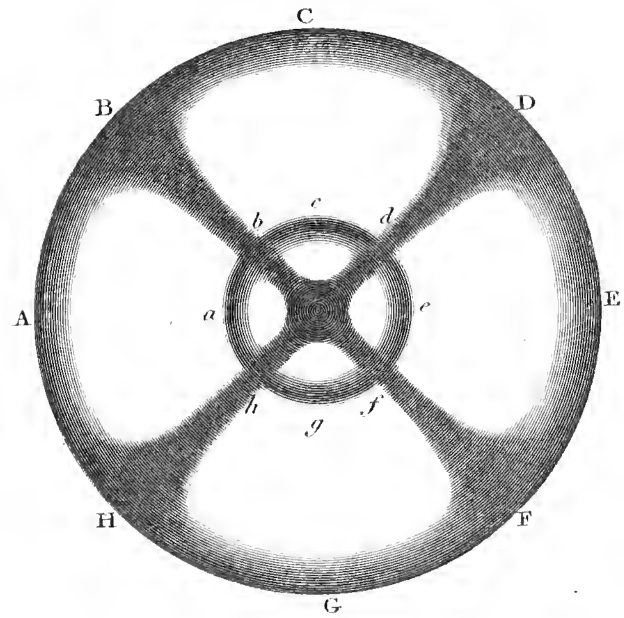

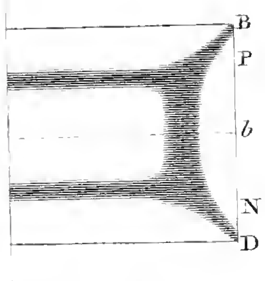

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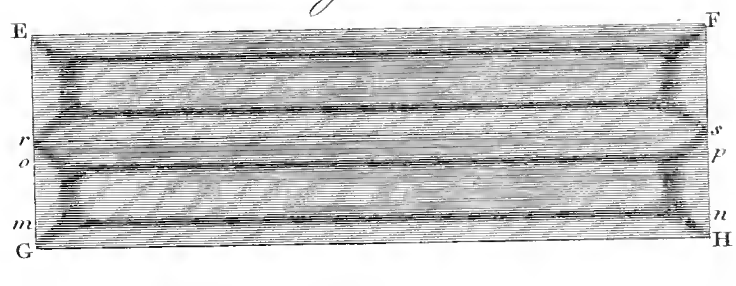

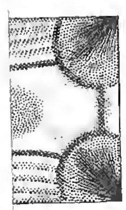

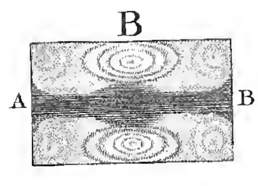

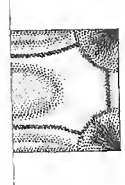

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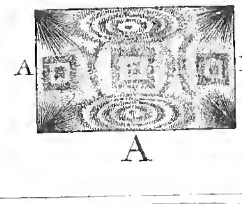

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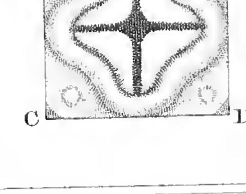

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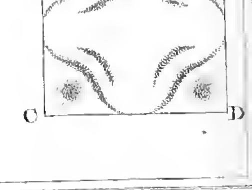

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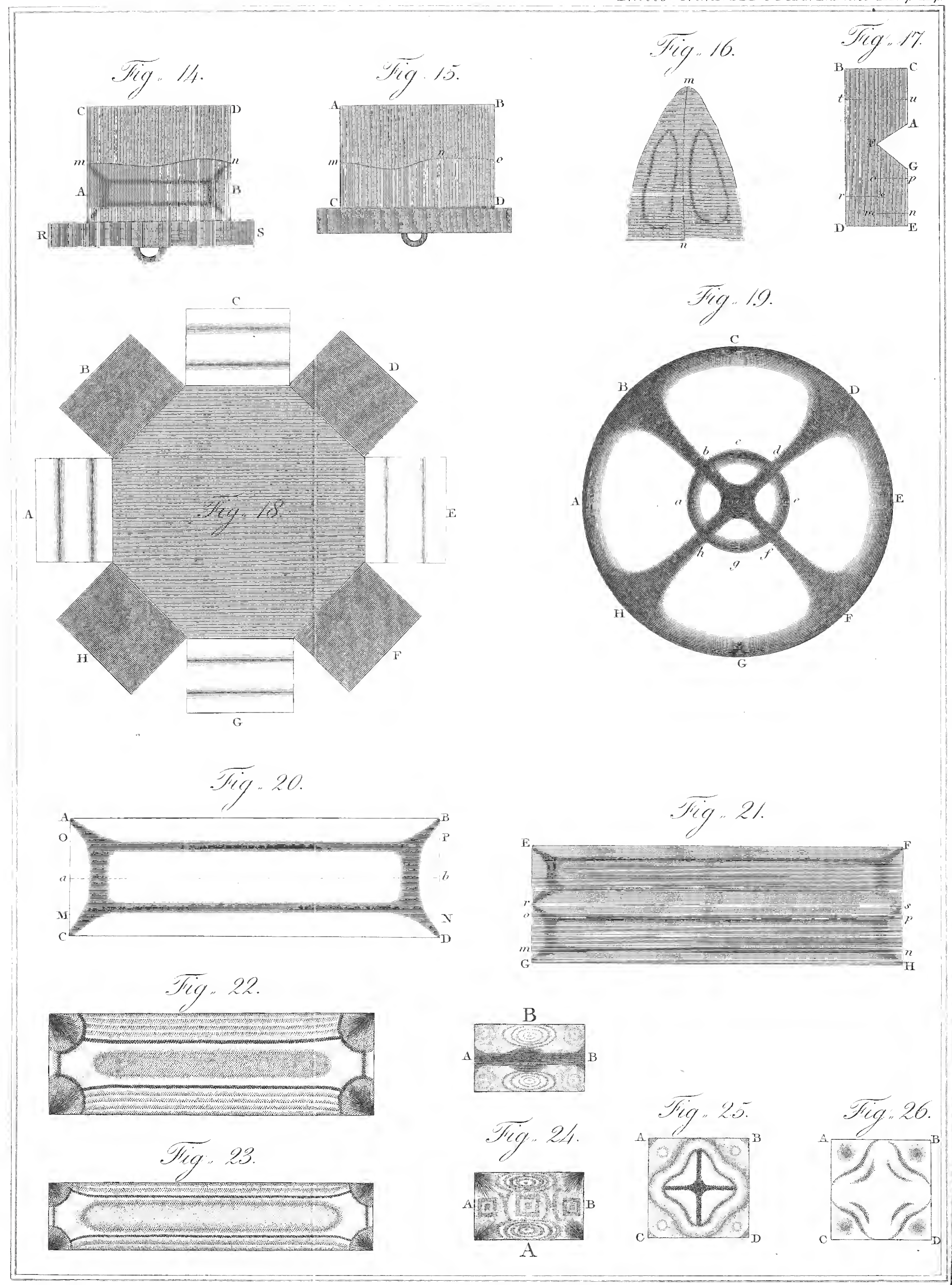

Fig. 14.
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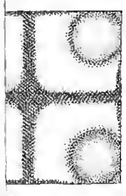

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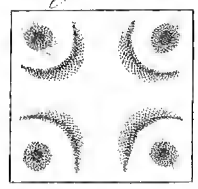

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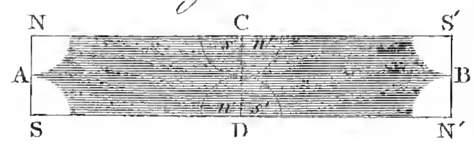

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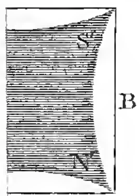

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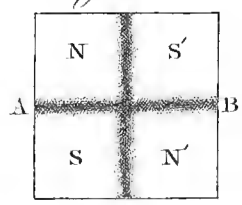

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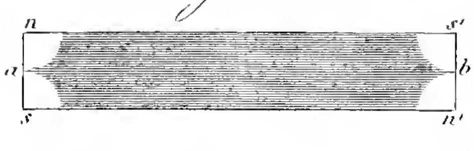

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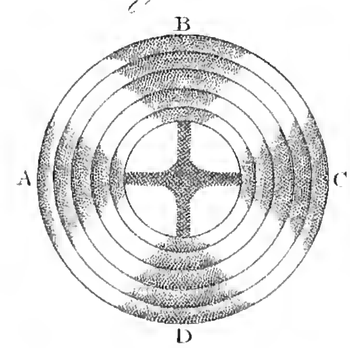

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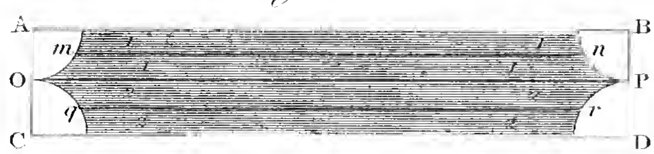

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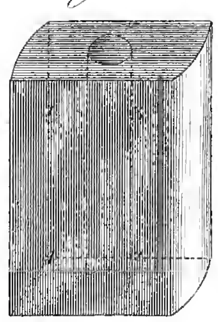

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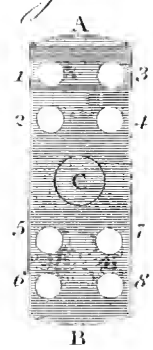

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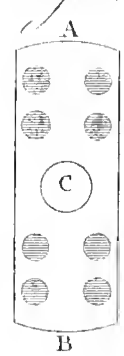

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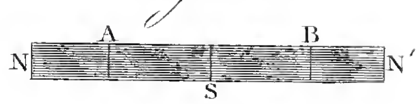

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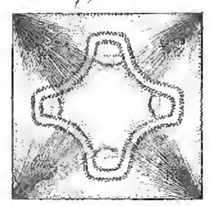

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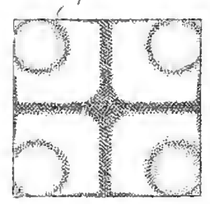

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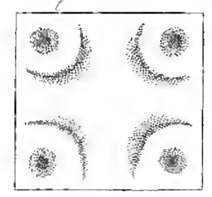

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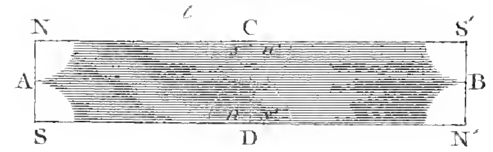

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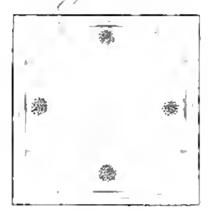

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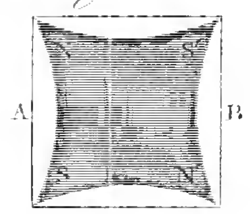

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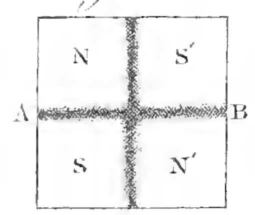

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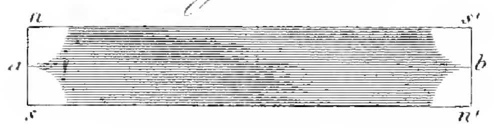

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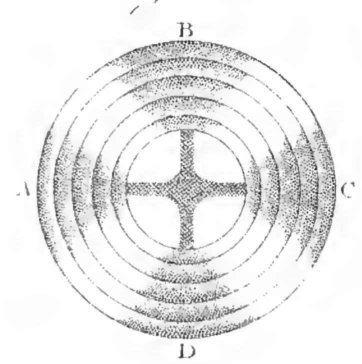

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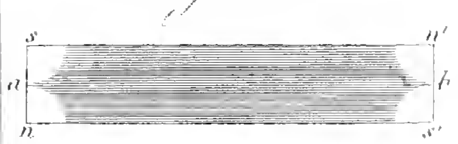

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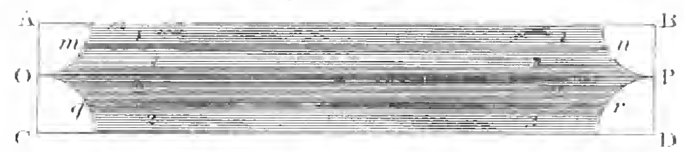

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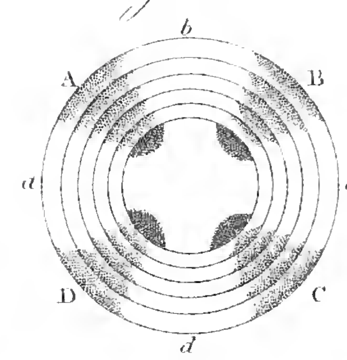

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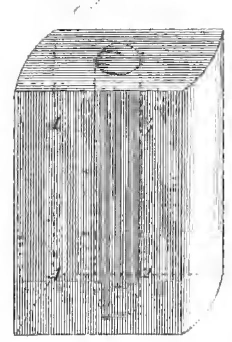

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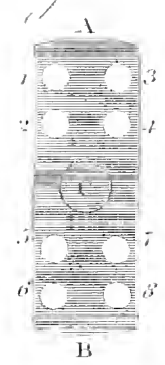

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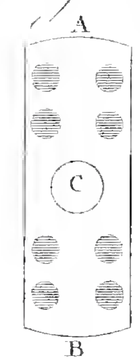

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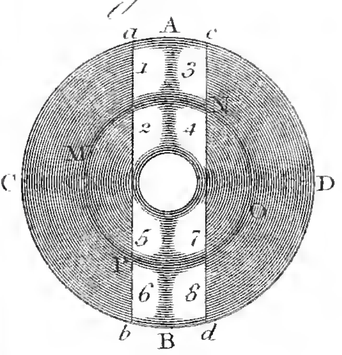

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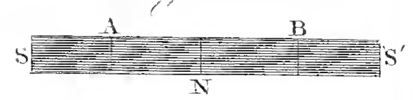

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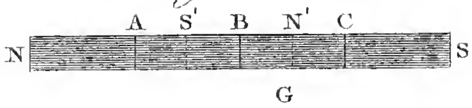

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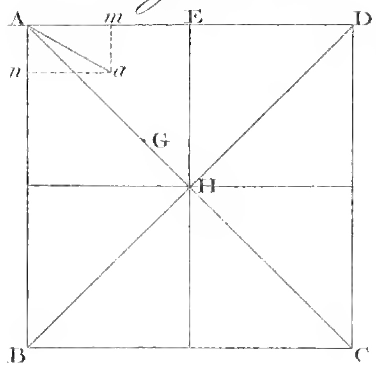

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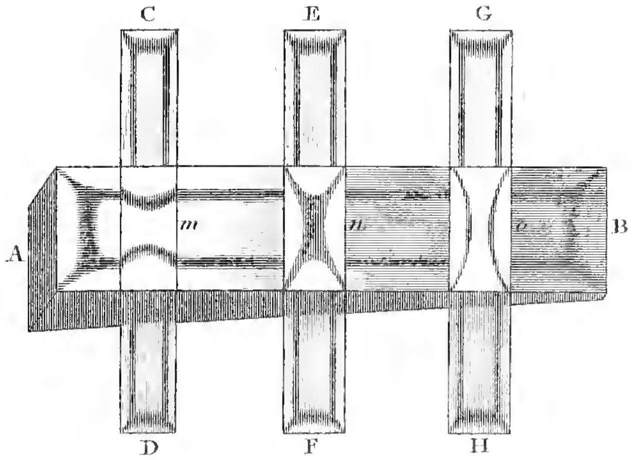


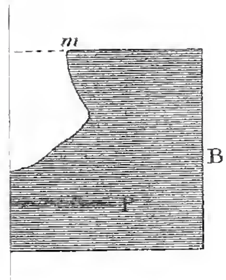
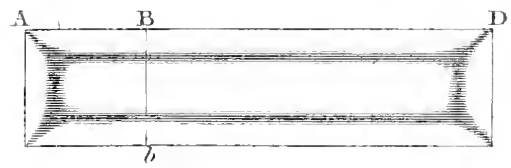


Fig. 54.



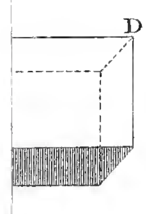


Fig. 50.

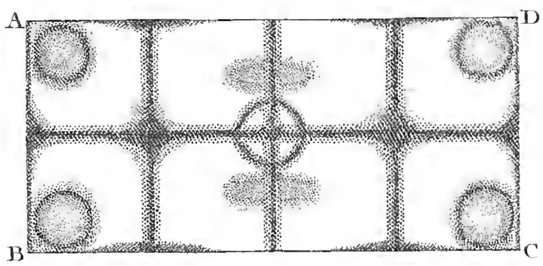
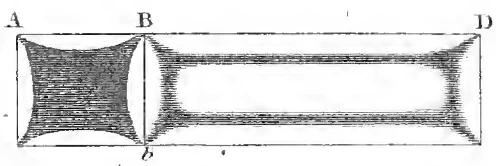


Fig. 53.



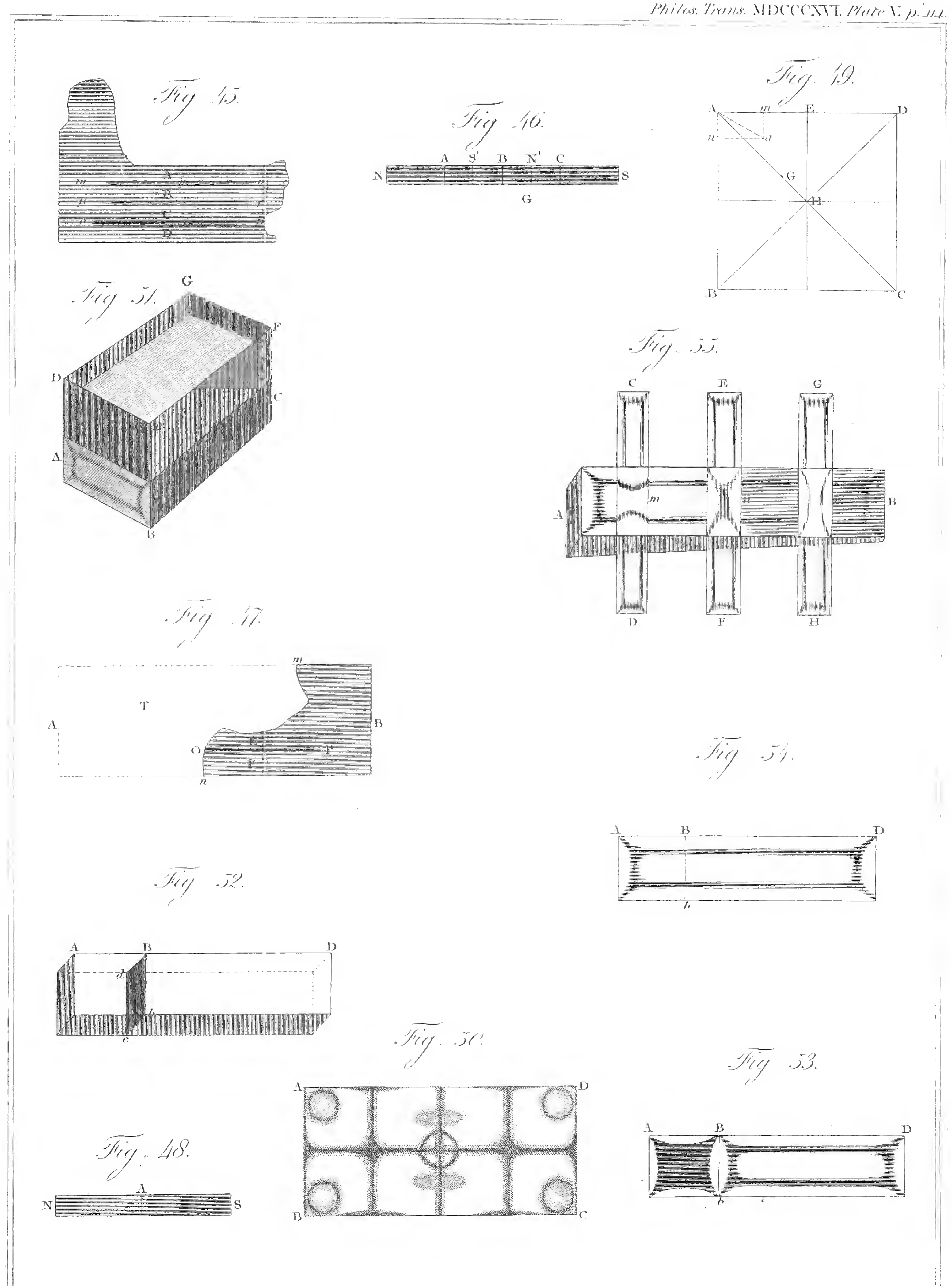


Fig. 45.
Fig. 46.
N A S' B N' C S
G
Fig. 49.
A m E D
n a
G
H
B C
Fig. 51.
G
F
D
C
E
A
B
Fig. 55.
C E G
A m n o B
D F H
Fig. 47.
m
A T B
O E P
F
n
Fig. 54.
A B D
b
Fig. 52.
A B D
d
b
c
Fig. 50.
A D
B C
Fig. 53.
A B D
b
Fig. 48.
N A S

J. Basire sc.

V. *Farther experiments on the combustion of explosive mixtures confined by wire-gauze, with some observations on flame.* *By Sir* H. Davy, *LL. D. F. R. S. V. P. R. I.*

Read January 25, 1816.

I HAVE pursued my enquiries respecting the limits of the size of the apertures and of the wire in the metallic gauze, which I have applied to secure the coal miners from the explosions of fire-damp. Gauze made of brass wire, $\frac{1}{50}$ of an inch in thickness, and containing only ten apertures to the inch, or 100 apertures in the square inch, employed in the usual way as a guard of flame, did not communicate explosion in a mixture of 1 part of coal gas and 12 parts of air, as long as it was cool, but as soon as the top became hot, an explosion took place.

A quick lateral motion likewise enabled it to communicate explosion.

Gauze made of the same wire, containing 14 apertures to the inch, or 196 to the square inch, did not communicate explosion till it became strongly red hot, when it was no longer safe in explosive mixtures of coal gas; but no motion that could be given to it, by shaking it in a close jar, produced explosion.

Iron wire gauze of $\frac{1}{40}$, and containing 240 apertures in the square inch, was safe in explosive mixtures of coal gas, till it became strongly red hot at the top.

Iron wire gauze of $\frac{1}{50}$, and of 24 apertures to the inch, or of 576 to the square inch, appeared safe under all circumstances in explosive mixtures of coal gas. I kept up a continual flame in a cylinder of this kind, 8 inches high and 2 inches in diameter, for a quarter of an hour, varying the proportions of coal gas and air as far as was compatible with their inflammation; the top of the cylinder, for some minutes, was strongly red hot, but though the mixed gas was passed rapidly through it by pressure from a gasometer and a pair of double bellows, so as to make it a species of blast furnace, yet no explosion took place.

I mentioned in my last communication to the Society, that a flame confined in a cylinder of very fine wire gauze, did not explode a mixture of oxygene and hydrogene, but that the gases burnt in it with great vivacity. I have repeated this experiment in nearly a pint of the most explosive mixture of the two gases; they burnt violently within the cylinder, but, though the upper part became nearly white hot, yet no explosion was communicated, and it was necessary to withdraw the cylinder to prevent the brass wire from being melted.

These results are best explained by considering the nature of the flame of combustible bodies, which, in all cases, must be considered as the combustion of an *explosive mixture* of inflammable gas, or vapour and air; for it cannot be regarded as a mere combustion at the surface of contact of the inflammable matter: and the fact is proved by holding a taper or a piece of burning phosphorus within a large flame made by the combustion of alcohol, the flame of the candle or of the phosphorus will

appear in the centre of the other flame, proving that there is oxygene even in its interior part.

The heat communicated by flame must depend upon its mass; this is shown by the fact that the top of a slender cylinder of wire-gauze hardly ever becomes dull red in the experiment on an explosive mixture, whilst in a larger cylinder, made of the same material, the central part of the top soon becomes bright red. A large quantity of cold air thrown upon a small flame, lowers its heat beyond the explosive point, and in extinguishing a flame by blowing upon it, the effect is probably principally produced by this cause, assisted by a dilution of the explosive mixture.

If a piece of wire-gauze sieve is held over a flame of a lamp or of coal gas, it prevents the flame from passing it, and the phenomenon is precisely similar to that exhibited by the wire-gauze cylinders; the air passing through is found very hot, for it will convert paper into charcoal; and it is an explosive mixture, for it will inflame if a lighted taper is presented to it, but it is cooled below the explosive point by passing through wires even red hot, and by being mixed with a considerable quantity of air comparatively cold. The real temperature of visible flame is perhaps as high as any we are acquainted with. Mr. TENNANT was in the habit of showing an experiment, which demonstrates the intensity of its heat. He used to fuse a small filament of platinum in the flame of a common candle; and it is proved by many facts, that a stream of air may be made to render a metallic body white hot, yet not be itself luminous.

A considerable mass of heated metal is required to inflame

even coal gas, or the contact of the same mixture with an extensive heated surface. An iron wire of $\frac{1}{20}$ of an inch and 8 inches long red hot, when held perpendicularly in a stream of coal gas, did not inflame it, nor did a short wire of one sixth of an inch produce the effect held horizontally; but wire of the same size, when six inches of it were red hot, and when it was held perpendicularly in a bottle, containing an explosive mixture, so that heat was successively communicated to portions of the gas, produced its explosion.

A certain degree of mechanical force which rapidly throws portions of cold explosive mixture upon flame, prevents explosions at the point of contact; thus on pressing an explosive mixture of coal gas from a syringe, or a gum elastic bottle, it burns only at some distance from the aperture from which it is disengaged.

Taking all these circumstances into account, there appears no difficulty in explaining the combustion of explosive mixtures within and not without the cylinders; for a current is established from below upwards, and the hottest part of the cylinder is where the results of combustion, the water, carbonic acid, or azote, which are not inflammable, pass out. The gas which enters is not sufficiently heated on the outside of the wire, to be exploded, and as the gases are no where confined, there can be no mechanical force pressing currents of flame towards the same point.

It will be needless to enter into further illustrations of the theoretical part of the subject: and I shall conclude this Paper by stating, what I am sure will be gratifying to the Society, that the cylinder lamps have been tried in two of the most

dangerous mines near Newcastle, with perfect success; and from the communications I have had from the collieries, there is every reason to believe that they will be immediately adopted in all the mines in that neighbourhood, where there is any danger from fire-damp.

VI. *Some observations and experiments made on the Torpedo of the Cape of Good Hope in the year* 1812. *By* John T. Todd, *late surgeon of His Majesty's ship Lion. Communicated by Sir* Everard Home, *Bart. V. P. R. S.*

Read February 15, 1816.

WHILST the Lion was stationed at the Cape of Good Hope, the seine, as is the custom throughout the navy, was frequently employed in procuring fish for the use of the ship's company, and besides the more edible kinds, many of the Torpedo were caught. In this manner the opportunity was afforded me of making the following observations, some of the imperfections of which I must be allowed to attribute to the "*manus nuda*" of my situation. The fish were generally caught early in the morning, and examined as soon after as possible. When this could not be done, they were placed in buckets of sea-water, where they sometimes remained alive for three, and in one instance for five days.

The torpedo is seldom met with to the eastward of the Cape of Good Hope. Hence, whilst I rarely failed in procuring them in Table Bay, I never but once succeeded in doing so in Simon's Bay, although the opportunities were the same in both places. It was never caught but by the seine, although the hook and line, with bait of every variety, were as often made use of exactly in the same situations. It differs in no respect, as far as I have been able to observe, from the same fish of the northern hemisphere, except that it was never

found so large; being never more than eight, nor less than five inches in length, and never more than five, nor less than three inches and a half in breadth. The colour of the animal is various; the upper surface being generally hazel grey, reddish brown, or purple; the under surface greyish white, yellowish white, or white with black patches.

The columns of the electrical organs were larger, and less numerous in proportion, than those described by Mr. HUNTER, in the torpedo caught at La Rochelle. When separate and uninfluenced by external pressure, they appear to be of the form of cylinders, as is shown as nearly as possible by suspending them by one of their extremities. The different forms which they exhibit in a horizontal section of the whole organ, are produced by their unequal attachment to one another by the intermediate reticular substance.

The electrical organs are so placed within the curvature of the semilunar cartilages of the large lateral fins, as to be entirely under the influence of the muscles, which are inserted into these cartilages. So that in any lateral motions of these cartilages towards the trunk, or in any increase of curvature of these cartilages, the electrical organs must be compressed. There appears also to be a muscular structure, which connects the anterior part of these cartilages to a process projecting from the anterior part of the cranium, the action of which must tend to increase this effect.

The inferior and posterior terminations of the small lateral fins are covered with laminæ of osseous matter, which are enveloped in the epidermis.

A much larger proportion of nerves is supplied to the electrical than to any other organs. This has appeared to others so

important an observation, that it may be repeated with propriety.

The shocks received from the torpedos which I examined, were never sensible above the shoulder, and seldom above the elbow-joint. The intensity of the shock bore no relation to the size of the animal (sensation being the only measure of intensity), but an evident relation to the liveliness of the animal, and *vice versâ.* The shocks generally followed simple contact, or such irritation as pressing, pricking, or squeezing, sometimes immediately, and sometimes not until after frequent repetition. Not unfrequently, however, animals apparently perfectly vivacious suffered this irritation without discharging any shock. There appeared no regularity of interval between the shocks. Sometimes they were so frequent as not to be counted; at other times, not more than one or two have been received from one animal; and, in a few instances, it has been impossible by any irritation to elicit shocks from some of them. When caught by the hand, they sometimes writhed and twisted about, endeavouring to extricate themselves by muscular exertion, and did not, until they found these means unavailing, discharge the shock. In many instances, however, they had recourse to their electrical power immediately.

The electrical discharge was, in general, accompanied by an evident muscular action. This was marked by an apparent swelling of the superior surface of the electrical organs, particularly towards the anterior part, opposite to the cranium, and by a retraction of the eyes. It was so evident, that when the animal was held in the hand of another person, I was often able to point out when he received the shock. In this,

however, I was also sometimes deceived; and I think I have received shocks (particularly when the animal has been debilitated, and the shocks weak,) without having been able to observe this muscular action.

Two of these animals, as nearly alike in every circumstance as possible, being each placed in a separate bucket of sea-water, from one of them frequent shocks were elicited by irritation, *viz.* simple contact, or pricking, &c.; the other was allowed to remain undisturbed. The former became languid, the intensity of its shocks diminished, and it soon died; the last shocks being received in a continued succession, producing pricking sensations never extending above the hand. The latter continued vivacious, and lived until the third day. This experiment was frequently repeated with the same results; and it might be observed, in general, where there was no direct comparison made, that those which parted with the shocks most freely soonest became languid, and died; and those which parted with them most reluctantly, lived the longest.

Two torpedos being placed exactly in the same circumstances as the last-mentioned, from one shocks were elicited until it became debilitated. It was then allowed to remain until the following day. When they were both examined, it was found that the animal from which no shocks had been previously received, discharged them very freely; but it was with the greatest difficulty that they could be procured from the other.

Having made an incision on each side of the cranium and gills of a lively torpedo, I pushed aside the electrical organs, so as to expose and divide their nerves. The animal was then placed in a bucket of sea-water. On examining it in

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about two hours afterwards, I found it impossible to elicit shocks from it by any irritation; but it seemed to possess as much activity and liveliness as before, and lived as long as those animals from which shocks had not been received, and which had not undergone this change.

Two of these animals being procured, the nerves of the electrical organs of one of them were divided after the manner above described. They were placed each in separate buckets of sea-water, and allowed to remain undisturbed. This was performed in the morning, and when examined in the evening, t was impossible to distinguish between the liveliness or activity of either.

Of two of these animals, the nerves of the electrical organs of one of them were divided. Being placed each in separate buckets of sea-water, they were both irritated as nearly alike as possible. From the perfect animal, shocks were received; after frequent repetition it became weak, and incapable of discharging the shock, and soon died. The last shocks were not perceptible above the second joint of the thumb, and so weak as to require much attention to observe them. From the other no shocks could be received; it appeared as vivacious as before, and lived until the second day. This experiment was frequently repeated with nearly the same results.

The nerves of one electrical organ only being divided in a lively torpedo, from which shocks had been previously received, on irritating the animal it was still found capable of communicating the shock. Whether there was any difference in the degree of intensity could not be distinctly observed. One electrical organ being altogether removed, the animal still continued capable of discharging the electrical shock.

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Having divided one of the nerves of each electrical organ in a torpedo, from which shocks had been previously received, I still found the animal capable, after this change, of communicating the shock.

Having introduced a wire through the cranium of a torpedo, which had been communicating shocks very freely, all motion immediately ceased, and no irritation could excite the electrical shock.

I never received a shock from a torpedo, when held by the extremities of the lateral fins or tail.

The preceding account appears to me to afford grounds for the following conclusions.

1. That the electrical discharge of this animal is in every respect a vital action, being dependent on the life of the animal, and having a relation to the degree of life and to the degree of perfection of structure of the electrical organs.

2. That the action of the electrical organs is perfectly voluntary.

3. That frequent action of the electrical organs is injurious to the life of the animal; and, if continued, deprives the animal of it. Is this only an instance of a law common to all animals, that by long continued voluntary action they are deprived of life? Whence is the cause of the rapidity with which it takes place in this instance? Or is it owing to the re-action of the shock on the animal?

4. That those animals, in which the nerves of the electrical organs are intersected, lose the power of communicating the shock, but appear more vivacious, and live longer than those in which this change has not been produced, and in which this power is exerted. Is the loss of the power of commu-

nicating the shock to be attributed to the loss of voluntary power over the organ? Does this fact bear any analogy to the effects produced by castration in animals?

5. That the possession of one organ only is sufficient to produce the shock.

6. That the perfect state of all the nerves of the electrical organs, is not necessary to produce the shock.

And, 7. From the whole it may be concluded, that a more intimate relation exists between the nervous system and electrical organs of the torpedo, both as to structure and functions, than between the same and any organs of any animal with which we are acquainted. And this is particularly shown, 1st, By the large proportion of nerves supplied to the electrical organs: and, 2d, By the relation of the action of the electrical organs to the life of the animal, and *vice versâ.*

VII. *Direct and expeditious methods of calculating the excentric from the mean anomaly of a planet. By the Reverend* Abram Robertson, *D. D. F. R. S. Savilian Professor of Astronomy in the University of Oxford, and Radcliffian Observer. Communicated by the Right Hon. Sir* Joseph Banks, *Bart. G. C. B. P. R. S.*

Read February 15, 1816.

SINCE the publication of KEPLER's discoveries in astronomy, the attention of men of science has frequently been directed to the problem distinguished by his name, and their exertions have frequently been employed to overcome the acknowledged difficulty of its solution. A statement of the various degrees of success, with which these endeavours have been made, is foreign to the present design. An account of this kind is now also needless, as Dr. BRINKLEY's examination of such attempts, published in the ninth volume of the Transactions of the Royal Irish Academy, affords a satisfactory review of most of the proceedings on this subject, previous to the year 1802.

After the following methods had occurred to my consideration, and I had fully proved their utility by actual application to examples, I was anxious to ascertain whether any author had anticipated me in the manner in which the investigations are conducted. With this view I examined such solutions as are referred to in Dr. BRINKLEY's very able Memoir, all those mentioned by MONTUCLA,* of which I could procure a sight,

* Histoire des Mathematiques, Tom. II. p. 343, &c. I have searched, without success, for Lorgna's and Trembley's publications.

and some others which had occurred to me in the course of my reading. The result of this search is a belief that no one before has aimed at a direct solution through the same small angle, and, by means of an equation in which this angle and its powers are the only unknown quantities, obtained a quickly converging series for its value in known terms. The small angle being found, with due precision, the eccentric anomaly readily becomes known.

M. DELAMBRE, in his Astronomy, published at Paris in 1814, in three quarto volumes, calculates the excentric anomaly by a method founded on those of CASSINI,* LA CAILLE,† SIMPSON,‡ and CAGNOLI.§ This eminent astronomer says of it, " Ce procédé, le plus directe que je connaises, est aussi le plus précis; il n'est qu'approximatif, mais il est toujours exact au-delà des dixièmes de seconde pour toutes les planètes de notre système." That the reader may readily judge how far the third method, which I now propose, deserves attention, I have annexed to my investigation the calculation by it of M. DELAMBRE's two examples, and also one relating to the comet of 1682 and 1759.

Each of the following methods of solution is to be considered as direct, although it proceeds through the medium of what is commonly called CASSINI's approximation. This approximation, as here used, can only be considered as the first certain step in the computation. No hypothesis is introduced

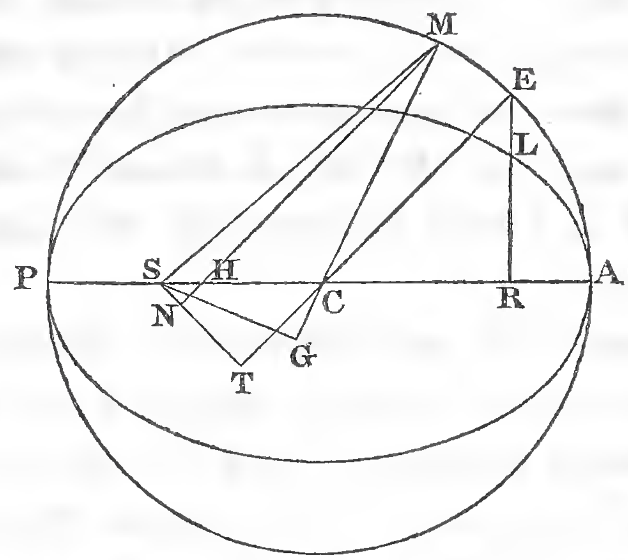
* Mémoires de l'Academie de l'année 1719.

† Leçons Elémentaires d'Astronomie, Paris, 1761.

‡ Essays on several curious and useful subjects, London, 1740.

§ Trigonométrie, Paris, 1786.

into the proceeding, and therefore no correction by trial and error is requisite.



Let ALP be the orbit of a planet, C the centre of the ellipse, S that focus in which the sun is placed, and AMP a circle described on the greater axis AP as a diameter. Let L be the true place of the planet, and AM the corresponding mean anomaly. Through L let the straight line ER be drawn, perpendicular to AP, and let it meet AP in R and the circle in E. Let EC, CM, SM be drawn, and let ST be perpendicular to EC, SG to CM and MN to ST. Then MN is parallel to ET, and NT is equal to the sine of the arc EM. It is easily proved, as in almost every writer on the subject, that ST is equal to the arc EM.

In this Problem it is supposed that AC, CS, AM are given, and it is required to find AE the excentric anomaly, for AE being found, the true anomaly ASL is easily obtained.

In each of the three following methods the angles CMS, CSM are used, and their difference is found by this proportion.

CM + CS : CM — CS : : tan. $\frac{1}{2}$ ACM : tan. $\frac{1}{2}$ (CSM — CMS). Hence the angles become known by their sum and difference. As the angle SMN is very small, and consequently the angle MSC nearly equal to ECA in the orbits of almost all the planets, this way of finding the angle CSM is usually called CASSINI'S approximation to the excentric anomaly ACE.

FIRST METHOD.

Having found the angle CSM, sin. CSM : CM : : sin. SCM : SM, which therefore becomes known. Let z equal the angle SMN, s equal the series expressing its sine, and c equal the series expressing its cosine. Put a equal the sine of CMS, and b equal its cosine. Then, radius being 1, $ac - bs =$ sin. CMN $=$ sin. ECM $=$ sin. (CMS $- z$). Also, 1 : SM : : s : SN $=$ SM $\times s$, and $ac - bs +$ SM $\times s =$ TN $+$ SN $=$ EM $=$ CMS$-z$, and therefore CMS $= z + ac - bs +$ SM $\times s = z + ac + \overline{\text{SM} - b} \times s$.

Let $d =$ SM $- b$, and then CMS $= z + ac + ds = z + a\left(1 - \frac{z^2}{2} + \frac{z^4}{2.3.4} - \frac{z^6}{2.3.4.5.6} + \frac{z^8}{2.3.4.5.6.7.8} - \&c.\right) + d\left(z - \frac{z^3}{2.3} + \frac{z^5}{2.3.4.5} - \&c.\right) = a + z + dz - \frac{az^2}{2} - \frac{dz^3}{2.3} + \frac{az^4}{2.3.4} + \frac{dz^5}{2.3.4.5} - \&c.$

Let $e =$ CMS$-a$, and putting A, B, C, &c. for the coefficients $e = Az - Bz^2 - Cz^3 + Dz^4 + Ez^5 - Fz^6 - Gz^7 + \&c.$

By reversing this series we find $z = \frac{1}{1+d}e + \frac{a}{2(1+d)^3}e^2 + \frac{d}{6(1+d)^4}e^3 + \frac{a^2}{2(1+d)^5}e^3 - \frac{a}{24(1+d)^5}e^4 + \frac{5ad}{12(1+d)^6}e^4 + \frac{5a^3}{8(1+d)^7}e^4 + \&c.$

This equation is in parts of the radius, and in order to have it in degrees we use this proportion, $1 : 57°.2957795 : : z : 57.2957795\,z = R° z$, putting R° for 57°.2957795.

Hence $R^\circ z = \frac{R^\circ}{1+d} e + \frac{R^\circ a}{2(1+d)^3} e^2 + \frac{R^\circ d}{6(1+d)^4} e^3 + \frac{R^\circ a^2}{2(1+d)^5} e^3 +$ &c.

SECOND METHOD.

The substitutions for the sines and cosines of the angles CMS, SMN being as in the preceding method, let SG be perpendicular to MC, and then radius being 1, 1 : CS : : sin. ACM : CS × sin. ACM = SG. But sin. CMS : SG : : sin. SMN : SN, that is, a : CS × sin. ACM : : s : $\frac{\text{CS} \times \text{sin. ACM}}{a} s$ = SN, and therefore $ac - bs + \frac{\text{CS} \times \text{sin. ACM}}{a} s$ = TN + SN = ST = CMS − z. Let $d = \frac{\text{CS} \times \text{sin. ACM}}{a} - b$, and then $ac + ds$ = CMS − z, and CMS = $z + ac + ds$, as in the preceding method.

THIRD METHOD.

Let z = the angle SMN, s = the series expressing its sine, and c = the series expressing its cosine, as before; but let a now denote the sine of CSM, and b its cosine, and let MN meet CS in H. Then $ac + bs$ is equal to the sine of the sum of the angles SMH, MSC, that is, $ac + bs$ = sin. MHA = sin. ACE, the excentric anomaly. We have therefore CE : $ac + bs$: : CS : $\frac{\text{CS}(ac+bs)}{\text{CE}}$ = ST = CMS − z.

Let $d = \frac{\text{CS} \times a}{\text{CE}}$, and $e = \frac{\text{CS} \times b}{\text{CE}}$, and then $dc + es$ = CMS − z, and CMS = $z + dc + es = z + d - \frac{dz^2}{2} + \frac{dz^4}{2.3.4} - \frac{dz^6}{2.3.4.5.6}$ + &c. + $ez - \frac{ez^3}{2.3} + \frac{ez^5}{2.3.4.5}$ − &c. = $d + z + ez - \frac{dz^2}{2} - \frac{ez^3}{2.3} + \frac{dz^4}{2.3.4} + \frac{ez^5}{2.3.4.5}$ − &c. Let f = CMS − d, and then $f = \text{A}z - \text{B}z^2 - \text{C}z^3 + \text{D}z^4 + \text{E}z^5$ − &c. putting A, B, C, &c. for the coefficients. By reversing this series, or by putting d for a,

e for d, and f for e, in the series in the first method, we find
$R^\circ z = \frac{R^\circ}{1+e} f + \frac{R^\circ d}{2(1+e)^3} f^2 + \frac{R^\circ e}{6(1+e)^4} f^3 + \frac{R^\circ d^2}{2(1+e)^5} f^3 - \frac{R^\circ d}{24(1+e)^5} f^4 + \frac{5R^\circ de}{12(1+e)^6} f^4 + \frac{5R^\circ d^3}{8(1+e)^7} f^4 +$ &c.

I prefer this method to the first or second, and therefore I proceed to calculate by it.

EXAMPLE I.

Let us suppose with M. DELAMBRE* that the mean anomaly is 135°, and the excentricity of the orbit 0.25, the mean distance from the sun being 1.

Here CM+CS=1.25, CM−CS= .75, and $\frac{CM-CS}{CM+CS} = \frac{.75}{1.25}$, the log. of which is to be used for any given mean anomaly in the orbit.

$\frac{.75}{1.25}$ - -	Log. 9.7781513	CMS is found by this proportion, 206264″.8 : 1 : : CMS in seconds : its length in parts of the radius.	
Log. tan. 67°..30′ -	10.3827757		
Log. tan. 55..22..49.84	10.1609270		
CSM = 122..52..49.84			
CMS = 12.. 7..10.16=43630″.16 -		Log. 4.6397867	
206264.8 -		Log. 5.3144251	
CMS = . 2115249		Log. 9.3253616	
CS = 0.25	Log. 9.3979400	CS - -	Log. 9.3979400
a	Log. 9 9241783	b - -	Log. 9.7347108
d=.2099511	Log. 9.3221183	e=.1357222	Log. 9.1326508.
.2115249 = CMS		As CSM is obtuse, e is negative.	
.0015738=f,	Log. 7.1969495	1+e=.8642778	Log. 9.9366534.

* Vol. II. page 28.

The angle SMN is therefore calculated from the series in the following manner:

First term.			Second term.		
R° - - -	Log.	1.7581226	R° - -	Log.	1.7581226
f - - - -	Log.	7.1969495	d - -	Log.	9.3221183
		8.9550721	$f.^2$ - -	Log.	4.3938990
$1+e$ - - -	Log.	9.9366534			15.4741399
Numb. - .1043322	Log.	9.0184187	2, Log. - -		0.3010300
For. 2d term .0000231					15.1731099
Sum - 0°.1043553 =		6′..15.″67	$(1+e)^3$ -	Log.	9.8099602
= SMH. CSM - =		122°..52..49. 84	Numb. .0000231	Log.	5.3631497
ACE = - -		122..59..5.51			

This differs from M. Delambre's conclusion only in the second place of the decimals.

Example II.

Supposing, with M. Delambre, that the mean anomaly in the same orbit is 96°, required the excentric anomaly.

We have as before	$\frac{.75}{1.25}$ - - - - - -	Log.	9.7781513
Log. tan.	48°.. - - - - - - - -		10.0455626
Log. tan.	33..40′..41″.51 - - - - - - -		9.8237139
CSM =	81..40..41″.51		
CMS =	14..19..18. 49 = 51558.49	Log.	4.7123003
	206264.8	Log.	5.3144251
	CMS = .2499627	Log.	9.3978752

CS = 0.25	- -	Log.	9.3979400
a	- - -	Log.	9.9954030
d = .2473677	-	Log.	9.3933430
.2499627 = CMS			
.0025950 = f	-	Log.	7.4141374

First term of the series.

R°	- - -	Log.	1.7581226
f	- - -	Log.	7.4141374
			9.1722600
$(1+e)$	- -	Log.	.0154366
Num.	.1434906	Log.	9.1568234
2d term	.0000429		
Sum	0°.1435335	=	8′..36″.72
= SMN. CSM	-	=	81°..40..41 .51
ACE	-	=	81..49..18 .23

CS = 0.25		Log.	9.3979400
b	- -	Log.	9.1605669
e = .0361832		Log.	8.5585069
$1+e$=1.0361832		Log.	.0154366

Second term of the series.

R°	- -	Log.	1.7581226
d	- -	Log.	9.3933430
f^2	- -	Log.	4.8282748
			15.9797404
2	- -	Log.	.3010300
			15.6787104
$(1+e)^3$		Log.	.0463098
Num. .0000429		Log.	5.6324006

This also differs from M. DELAMBRE's conclusion only in the second place of decimals.

EXAMPLE III.

Let us suppose ALP to be the orbit of the comet which appeared in 1682, and reappeared in 1759, according to the prediction of Dr. HALLEY; that CE is equal to 18.07575, that that CS is equal to 17.49225, and that the mean anomaly is 179°..47′..32″.17, it is required to find the excentric anomaly.

Here $\frac{CM-CS}{CM+CS} = \frac{.5835}{35.568}$		- - -	Log.	8.2149815
Log. tan.	89°..53′..46″.08	- -		12.7415531
Lon. tan.	83..41..34 .5	- -		10.9565346
CSM =	173..35..20 58			
CMS =	6..12..11 58 = 22331.58		Log.	4.3489194
	206264.8		Log.	5.3144251
	CMS = .1082666		Log.	9.0344943

CS = 17.49225		Log.	1.2428457
a	- -	Log.	9.0478932
			10.2907389
CE = 18.07575		Log.	1.2570963
d = .1080544		Log.	9.0336426
.1082666 = CMS			
.0002122 = f,		Log.	6.3267454

First term of the series.

R°	- -	Log.	1.7581226
f	- - -	Log.	6.3267454
			8.0848680
$(1+e)$	- -	Log.	8.5835638
Num. .3171788		Log.	9.5013042

Second term of the series.

R°	- -	Log.	1.7581226
f^2	- -	Log.	2.6534908
d	- - -	Log.	9.0336426
			13.4452560
2	- -	Log.	.3010300
			13.1442260
$(1+e)^3$	-	Log.	5.7506914
Num. .0024748		Log.	7.3935346

.0024748
.3171788
.0000386

.3196922 sum of the positive
— .0000406

CS = 17.49225		Log.	1.2428457
b	- -	Log.	9.9972757
			11.2401214
CE=18. 07575		Log.	1.2570963
e = . 9616678		Log.	9.9830251

As CSM is obtuse, e is negative.

$1+e$=.0383322		Log.	8.5835638

Third term of the series.

R°	- -	Log.	1.7581226
e	- -	Log.	9.9830251
f^3	- -	Log.	8.9802362
			20.7213839
6	- -	Log.	.7781513
			19.9432326
$(1+e)^4$	-	Log.	4.3342552
Num. — .0000406		Log.	5.6089774

Fourth term of the series.

R°	- -	Log.	1.7581226
d^2	- -	Log.	8.0672852
f^3	- -	Log.	8.9802362
			18.8056440
2	- -	Log.	.3010300
			18.5046140
$(1+e)^5$	- -	Log.	2.9178190
Num. .0000386		Log.	5.5867950

SMN=0°.3196516 = 19′..10″.75
CSM = 173..35..20 .58

173..54..31 .33 = ACE. With the design already

expressed, I adopted in this example the same data with Mr. IVORY: see Transactions of the Royal Society of Edinburgh, Vol. V. page 236.

The preceding method bears a nearer resemblance to that given by KEILL, in his Astronomy, than to any other of which I know. Adapting his manner of proceeding to the figure here used, he puts $y =$ EM, $e =$ sin. AM, f cos. AM, and $g =$ CS. The series expressing the sine of AE, is therefore equal to $e - fy - \frac{ey^2}{2} + \frac{fy^3}{2.3} + \frac{ey^4}{2.3.4}$ &c. But the radius, which is 1, is to the sine of AE as CS or g is to ST or EM, that is to y. Consequently $y = ge - gfy - \frac{gey^2}{2} + \frac{gfy^3}{2.3} + \frac{gey^4}{2.3.4}$ &c. and therefore $ge = y + gfy + \frac{gey^2}{2} - \frac{gfy^3}{2.3} - \frac{gey^4}{2.3.4}$ &c.

By reversing this he obtains a series, which, omitting the numbers in the coefficients, converges as the powers of $\frac{ge}{1+gf}$ or $\frac{\text{CS} \times \text{sin. AM}}{1 + \text{CS} \times \text{cos. AM}}$. This degree of convergency to the value of y in the foregoing examples is as follows.

In the first example as the powers of .2147372,
In the second example as the powers of .2553020,
In the third example as the powers of .1086700.

In the third method which has been here investigated, the series converges to the value of SMH as the powers of $\frac{f}{1+e}$ or $\frac{\text{CMS} - \text{CS} \times \text{sin. CSM}}{1 + \text{CS} \times \text{cos. CSM}}$. This degree of convergency in the foregoing examples is as follows.

In the first example as the powers of .0018209,
In the second example as the powers of .0250438,
In the third example as the powers of .0055358.

The third method, by which the three foregoing examples

are calculated, appears to me the most simple and precise in theory, and the most expeditious in practice of any which I have seen. This I say with the greater degree of freedom, as I am so well aware of the similarity between its series and that of KEILL's, and so perfectly convinced of the advantages which it derives from CASSINI's approximation, that I consider it, with the exception of some deviations, as a combination of their methods.

VIII. *Demonstrations of the late Dr.* Maskelyne's *formulæ for finding the longitude and latitude of a celestial object from its right ascension and declination; and for finding its right ascension and declination from its longitude and latitude, the obliquity of the ecliptic being given in both cases. By the Rev.* Abram Robertson, *D.D. F.R.S. Savilian Professor of Astronomy in the University of Oxford, and Radcliffian Observer. Communicated by the Right Honourable Sir* Joseph Banks, *Bart. G.C.B. P.R.S.*

Read February 15, 1816.

THE methods given by our late Astronomer Royal, for solving the two problems alluded to, were printed in his introduction to Taylor's Logarithmic Tables. Since their appearance before the public, they have met with the warmest approbation from those most capable of judging of their merit; but no one, so far as I know, has fully demonstrated them; nor has any one, so far as my knowledge extends, observed two mistakes with which they are accompanied, and which in certain cases would affect the accuracy of their application.

These circumstances, and a consideration of the high character of the author of the formulæ, induced me to reduce the following demonstrations and remarks into the form of a short memoir. I trust I shall not be charged with any improper motive for thus noticing the mistakes. Candour, I

hope, will view them only as accidental oversights, and the most sincere regard for his memory will allow the propriety of correcting them.

PROBLEM I.

"The right ascension and declination of a celestial object, together with the obliquity of the ecliptic, being given, to find its longitude and latitude."

Let QAR, Fig. 1, 2, 3, 4, 5, 6, (Pl. VI.) be the equator, P its north, and *p* its south pole. Let CAL be the ecliptic, E its north and *e* its south pole. In the first three figures, let P*p*R be the first and P*p*Q the fourth quadrant of right ascension; E*e*L the first and E*e*C the fourth quadrant of longitude, A in these figures being the first point of aries. In the last three figures, let P*p*R be the second and P*p*Q the third quadrant of right ascension; E*e*L the second and E*e* C the third quadrant of longitude, A in these figures being the first point of libra.

Let S be a celestial object, and let PSH or *p*SH be a circle of declination, and ESF or *e*SF a circle of latitude passing through it, the angle LAR or QAC being the obliquity of the ecliptic. Then, reckoning from the first point of aries and according to the order of the signs, AH is the right ascension, SH the declination, AF the longitude, and SF the latitude of S.

In each of the figures, let it be supposed that the arc of a great circle passes from A to S, and then SAH, SAF will be two right angled triangles.

By trigonometry, sin. AH : R :: tan. HS : tan. HAS $= \frac{\text{R tan. HS}}{\text{sin. AH}} = \frac{\text{R tan. declination}}{\text{sin. Æ.}}$, north or south as the declination is. Let this first auxiliary angle be called A, and let O denote the

obliquity of the ecliptic. Then in the first and second quadrants of right ascension, for a star whose declination is north, $A \sim O = SAF = B$, the second auxiliary angle, but in these quadrants for a star whose declination is south $A + O = SAF = B$.

In the third and fourth quadrants of right ascension, for a star whose declination is north, $A + O = SAF = B$, but in these quadrants for a star whose declination is south, $A \sim O = SAF = B$.

If S be on P*p*, as represented in Fig. 3 and 6, then $90° - O = SAF = B$.

To find the longitude,

We have the following proportions cos. SAH : R : : tan. AH : tan. SA, and R : cos. SAF : : tan. SA : tan. AF.

Hence cos. SAH : cos. SAF : : tan. AH : tan. AF $= \frac{\text{cos. SAF tan. AH}}{\text{cos. SAH}}$.

That is tan. longitude $= \frac{\text{cos. B tan. Æ}}{\text{cos. A}}$.

Or, as tan. A : R :: sin. A : cos. A $= \frac{\text{R sin. A}}{\text{tan. A}}$, this being put for cos. A in the preceding expression, we have also tan. longitude $= \frac{\text{tan. A cos. B tan. Æ}}{\text{R sin. A}}$.

If S be on P*p*, then R : cos. SAF : : tan. SA : tan. AF $= \frac{\text{cos. SAF tan. SA}}{\text{R}}$. That is tan. longitude $= \frac{\text{cos. (90°—O) tan. declin.}}{\text{R}}$.

To find the latitude.

By trigonometry, sin. AF : R :: tan. SF : tan. SAF, and therefore tan. SF $= \frac{\text{sin. AF tan. SAF}}{\text{R}}$, that is tan. latitude $= \frac{\text{sin. longitude tan. B}}{\text{R}}$.

But tan. AF : R :: sin. AF : cos. AF, and sin. AF $= \frac{\text{tan. AF cos. AF}}{\text{R}}$, and this being put in the preceding expression for sin. AF, we have also tan. latitude $= \frac{\text{tan. AF cos. AF tan. B}}{\text{R}^2} = \frac{\text{tan. long. cos. long. tan. B}}{\text{R}^2}$.

Rules for ascertaining the longitude from the preceding formulæ.

1. The longitude falls in the first, second, third, or fourth quadrant, according as the right ascension is in the first, second, third, or fourth quadrant, unless the auxiliary angle B be equal to or greater than 90°.

2. If B be equal to 90° the longitude $= o$, if the right ascension is in the first or fourth quadrant; but if the right ascension is in the second or third, the longitude $=$ 180°.

3. If B be greater than 90° the following are the consequences. If the right ascension is in the first quadrant, the longitude falls in the fourth, and on the contrary, if the right ascension is in the fourth, the longitude falls in the first. If the right ascension is in the second, the longitude falls in the third, and on the contrary, if the right ascension is in the third, the longitude falls in the second.

4. If S be on the equinoctial colure, as represented in Fig. 3 and 6, (Pl. VI.) the following are the consequences. If S be between the first point of aries and P the longitude falls in the first quadrant, but if S be between the first point of libra and P the longitude falls in the second. If S be between the first point of aries and *p* the longitude falls in the fourth quadrant, but if S be between the first point of libra and *p* the longitude falls in the third quadrant.*

The first of these rules will be evident after the second and third are demonstrated.

* No provision is made in Dr. Maskelyne's formulæ for ascertaining the longitude of a celestial object on P*p* in either of the two hemispheres.

Demonstration of the second rule.

It is evident that the circle of latitude for any star in EA*e* coincides with EA*e*, and therefore in Fig. 1, 2, 3, (Pl. VI.) in which A represents the equinoctial point of aries, the longitude of such a star is *o*. Now in Fig. 3. let S be a star at the intersection of the arcs A*e*, *p*H, and in this case, SAL in the first quadrant is equal to B=90°. Again in Fig. 3. let S be a star at the intersection of the arcs EA, PH, and in this case SAC in the fourth quadrant is equal to B=90°. In Fig. 6. (Pl. VI.) let S be a star at the intersection of the arcs *e*A, *p*H, and according to the rule, SAL in the second quadrant is equal to B=90°. Lastly, in Fig. 6. let S be a star at the intersection of the arcs EA, PH, and according to the rule, SAC in the third quadrant is equal to B=90°. It follows from these circumstances, that if B be equal to 90°, the star must be in EA*e*, and therefore that its longitude must be either o or 180°.

Demonstration of the third Rule.

Let S be a star in Fig. 2. between the arcs A*p*, A*e*, and then it is evident that its right ascension H is in the first, but its longitude F is in the fourth quadrant, and that SAL=B is greater than *e*AL or 90°. Again let S be a star in Fig. 2. between the arcs EA, PA, and then it is evident that its right ascension H is in the fourth quadrant, but its longitude F is in the first, and SAC, which is equal to B, is greater than EAC or 90°. In Fig. 5. (Pl. VI.) let S be a star between the arcs A *e*, A*p*. Then H the right ascension is in the second quadrant, but F the longitude is in the third, and SAL, equal to B, is greater than *e* AL or 90°. Again in Fig. 5. let S be a star between the arcs EA, PA, and then it is evident that H

its right ascension is in the third quadrant, but F its longitude is in the second, and SAC, which is equal to B, is greater than EAC or 90°.

Hence it follows, that if B be greater than 90°, the star must be situated between *e* A and *p* A, or between EA and PA, and that the consequences with respect to its longitude, must be as stated in the third Rule.

Dr. MASKELYNE says, p. 59, Problem XIII. "Longitude will be of the same kind, or in the same quadrant of the circle as Æ is, unless B exceeds 90°, *which can only happen when* Æ *is in second semicircle.** Then if Æ be in third quadrant or from 6^s to 9^s, longitude will be in second quadrant or from 3^s to 6^s, and the operation will give L. cot. excess of long. above 3^s. Or if Æ be in fourth quadrant, or from 9^s to 12^s, longitude will be in first quadrant; and the operation will give, L.*t*, long. under 3^s, or in first quadrant."

PROBLEM II.

"The longitude and latitude of a celestial object, with the obliquity of the ecliptic, being given, to find its right ascension and declination."

Using the same figures as in the last article, by trigonometry, sin. AF : R :: tan. SF : tan. SAF $= \frac{\text{R tan. SF}}{\text{sin. AF}} = \frac{\text{R tan. latitude}}{\text{sin. longitude}}$, north or south as the latitude is.

Let this first auxiliary angle be called A. Then when the longitude is in the first or second quadrant $A+O=SAH=B$, the second auxiliary angle, if the latitude is north, but $A \sim O = SAH = B$, if the latitude is south.

* The words printed in italics contain a mistake, which would affect the longitude of any celestial object situated between *Ap*, *Ae*, and *pe*, both in Fig. 2 and 5.

When the longitude is in the third or fourth quadrant, then $A \sim O = SAH = B$, if the latitude is north, but $A + O = SAH = B$, if the latitude is south.

If S be on E*e*, as represented in Fig. 3, and 6, then $90° - O = SAH = B$.

To find the right ascension,

We have the following proportions, cos. SAF : R :: tan. AF : tan. SA and R : cos. SAH :: tan. SA : tan. AH

Hence, cos. SAF : cos. SAH :: tan. AF : tan. AH $= \frac{\text{cos. SAH tan. AF}}{\text{cos. SAF}}$.

That is tan. Æ $= \frac{\text{cos. B tan. longitude}}{\text{cos. A}}$.

But tan. A : R : : sin. A : cos. A $= \frac{\text{R sin. A}}{\text{tan. A}}$, and this being put for the cos. A in the preceding expression, we have also

Tan. Æ $= \frac{\text{tan. A cos. B tan. longitude}}{\text{R sin. A}}$.

If S be on E*e*, then R : cos. SAH : : tan. SA : tan. AH $= \frac{\text{cos. SAH tan. SA}}{\text{R}}$. That is tan. Æ $= \frac{\text{cos. } (90° - O) \text{ tan. latitude}}{\text{R}}$.

To find the declination.

By trigonometry sin. AH : R :: tan. SH : tan. SAH, and tan. SH $= \frac{\text{sin. AH tan. SAH}}{\text{R}}$; that is tan. declination $= \frac{\text{sin. Æ tan. B}}{\text{R}}$. But tan. AH : R :: sin. AH : cos. AH, and sin. AH $= \frac{\text{tan. AH cos. AH}}{\text{R}}$, and this being put for sin. AH in the preceding expression, we have also tan. declination $= \frac{\text{tan. Æ cos. Æ tan. B}}{\text{R}^2}$.

Rules for ascertaining the right ascension from the preceding formulæ.

1. The right ascension falls in the first, second, third or fourth quadrant, according as the longitude is in the first, second, third or fourth quadrant, unless the auxiliary angle B be equal to, or greater than 90°.

2. If B be equal to 90°, the right ascension is $= o$, if the longitude is in the first or fourth quadrant; but if the longitude is in the second or third, the right ascension is 180°.

3. If B be greater than 90°, the following are the consequences. If the longitude is in the first quadrant, the right ascension falls in the fourth, and on the contrary, if the longitude is in the fourth, the right ascension falls in the first. If the longitude is in the second, the right ascension falls in the third, and on the contrary, if the longitude is in the third, the right ascension falls in the second.

4. If S be on E*e*, as represented in Fig. 3 and 6, the following are the consequences. If S be between E and the first point of aries the right ascension falls in the fourth quadrant, but if S be between E and the first point of libra the right ascension falls in the third. If S be between *e* and the first point of aries the right ascension falls in the first quadrant, but if S be between *e* and the first point of libra the right ascension falls in the second quadrant.*

The first of these rules will be evident after the second and third are demonstrated.

Demonstration of the second Rule.

It is evident that the circle of declination for any star in PA*p*, coincides with PA*p*, and therefore in Fig. 1, 2, 3, (Pl. VI.) in which A represents the equinoctial point of aries, the right ascension of such a star is *o*. But in Fig. 4, 5, 6, (Pl. VI.) in which A represents the equinoctial point of libra, the right ascension of a star on PA*p* is 180°.

Now in Fig. 3. let S be a star at the intersection of the arcs

* No provision is made in Dr. Maskelyne's Formulæ for ascertaining the right ascension of a celestial object on E*e* in either of the two hemispheres.

PA, EF, and in this case SAR, in the first quadrant, is equal to B=90°. Again in Fig. 3. let S be a star at the intersection of the arcs *p* A, *e* F, and in this case SAQ, in the fourth quadrant = 90° = B. In Fig. 6. let S be a star at the intersection of the arcs P A, E F, and in this case SAR, in the second quadrant, = 90° = B. Lastly, in Fig. 6. let S be a star at the intersection of the arcs *p*A, *e*F, and in this case SAQ, in the third quadrant, = 90°=B. Hence it follows that if B = 90°, the star must be in PA*p*, and its right ascension as stated in the rule.

Demonstration of the third Rule.

In Fig. 2. let S be a star between the arcs EA, PA, whose longitude F is in the first quadrant, but its right ascension H in the fourth, and then according to the rule SAR, which is greater than PAR or 90° is equal to B. Again in Fig. 2. let S be a star between the arcs *p*A, *e*A, whose longitude F is in the fourth quadrant, but its right ascension H in the first, and then according to the rule, SAQ in the fourth, which is greater than *p*AQ or 90°, is equal to B. In Fig. 5. let S be a star between the arcs PA, EA, whose longitude F is in the second quadrant, but its right ascension H in the third, and then according to the rule, SAR which is greater than PAR or 90° is equal to B. Lastly, in Fig. 5. let S be a star between the arcs *p*A, *e*A, whose longitude F is in the third quadrant, but its right ascension H in the second, and then, according to the rule, SAQ, which is greater than *p*AQ, or 90°, is equal to B.

It therefore follows, that if B be greater than 90° the celestial object must be situated between EA and PA, or between *e*A and *p*A: for if it be not so situated, B will not be greater

than 90°. The consequences therefore with respect to its right ascension, must be as stated in the third rule.

In Dr. MASKELYNE'S XIVth Problem, it is said, " right ascension will be of the same kind, or in the same quadrant of the circle as the longitude is, unless B exceeds 90°, *which can only happen when long. is in 1st semicircle.** Then if long. be in 1st. quadrant; Æ will be in 4th quadrant; and the operation will give log. cot. excess of Æ above 9ˢ. Or if long. be in 2d. quadrant, Æ will be in 3d. quadrant, and the operation will give L. *t.* excess of Æ above 6ˢ.

In each of the two Problems the quantity which comes out by calculation, either for the longitude or right ascension, is the distance from the nearest equinoctial point. In the first quadrant this quantity itself is the longitude or right ascension. In the second quadrant this quantity must be subtracted from 180°, but in the third quadrant it must be added to 180°, and the difference or sum will be the longitude or right ascension sought. In the fourth quadrant this quantity must be subtracted from 360°, and the remainder will be the longitude or right ascension required.

M. DELAMBRE has duly appreciated the value of Dr. MASKELYNE'S method, while comparing it with that of M. LALANDE. LALANDE uses the four following proportions for finding the longitude and latitude.†

R : cos. AH : : cos. SH : cos. SA.
R : sin. AH : : cot. SH : cot. SAH.
R : cos. SAF : : tan. SA : tan. AF.
R : sin. SA : : sin. SAF: sin. SF.

* These words printed in italics also contain a mistake, similar to that pointed out in the preceding Problem. In consequence of these mistakes, in each of the two instances, the remarks after the words in italics in the quotations are incomplete.

† Page 304. Vol. I.

He afterwards observes that the right ascension and declination may be found from the longitude and latitude by means of the same analogies, by putting the longitude instead of right ascension, and latitude instead of declination.

Of these and Dr. MASKELYNE's formulæ, Mr. DELAMBRE proceeds to say,* " MASKELYNE a réduit à trois les quatre analogies de LALANDE. Par ce changement MASKELYNE a remédié fort heureusement à un défaut assez considérable de la méthode de LALANDE. Quand l'astre est voisin des points équinoxiaux, la première analogie de LALANDE qui fait trouver l'inconnue par son cosinus, ne peut donner aucune précision. MASKELYNE, au contraire, en évitant cette inconnue, qui n'est qu'un arc subsidiaire, n'emploie que la tangente, qui n'est jamais sujette à cet inconvénient."

* Page 494. Vol. I.

Fig. 1.

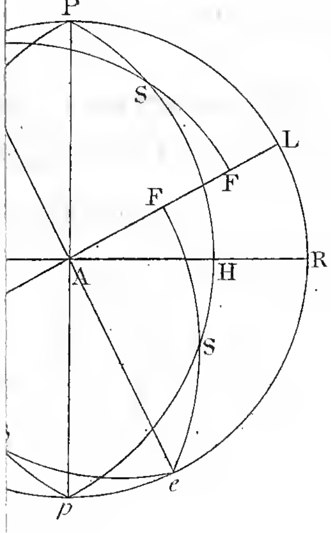


Fig. 4.

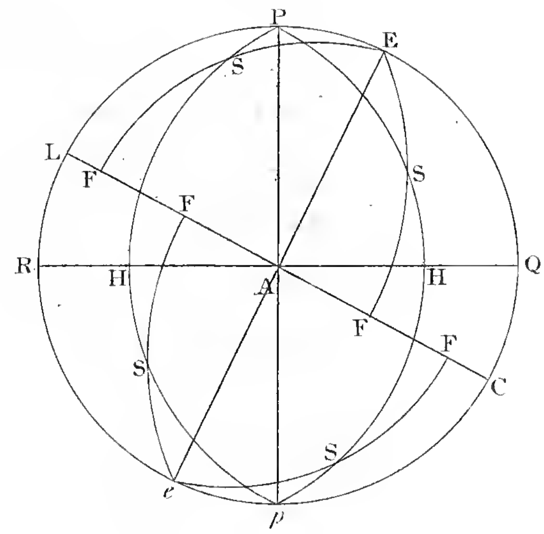


Fig. 2.

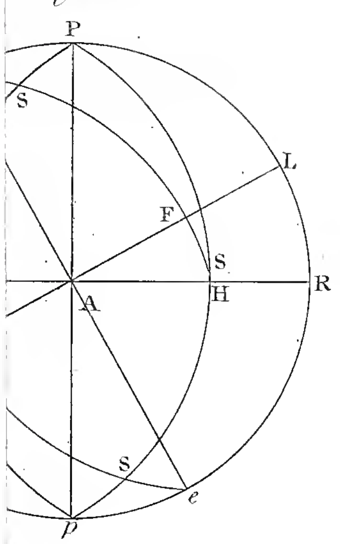


Fig. 5.

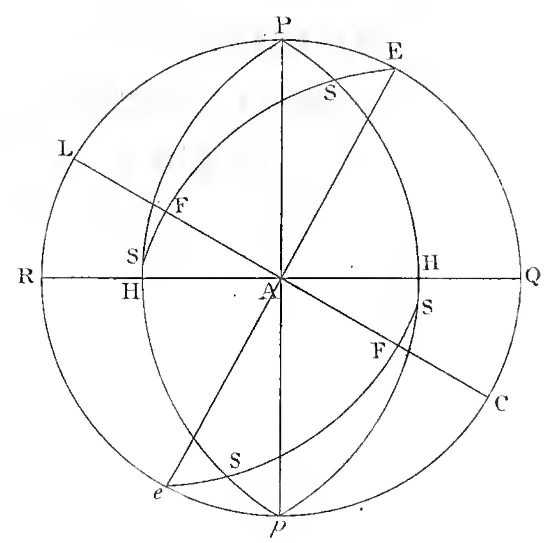


Fig. 3.

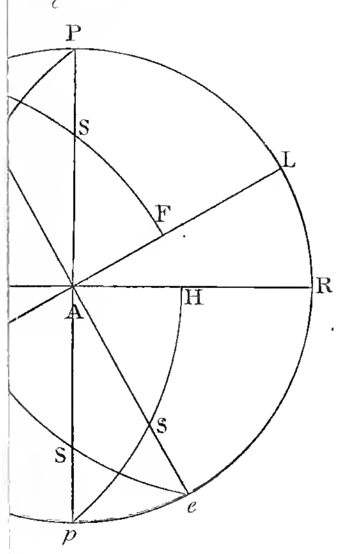
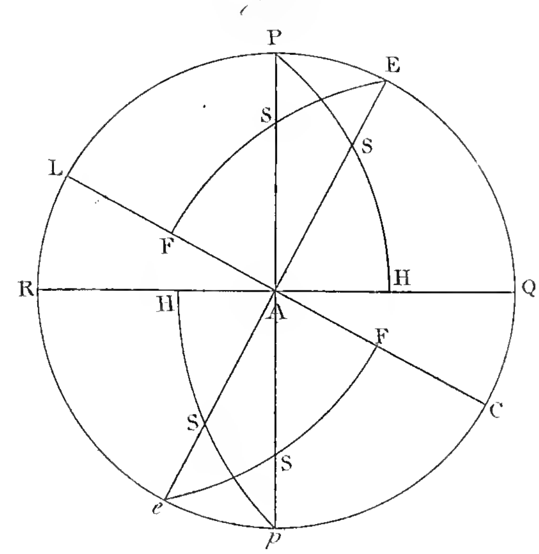


Fig. 6.



Jas. Basire, sc.

Fig. 1.

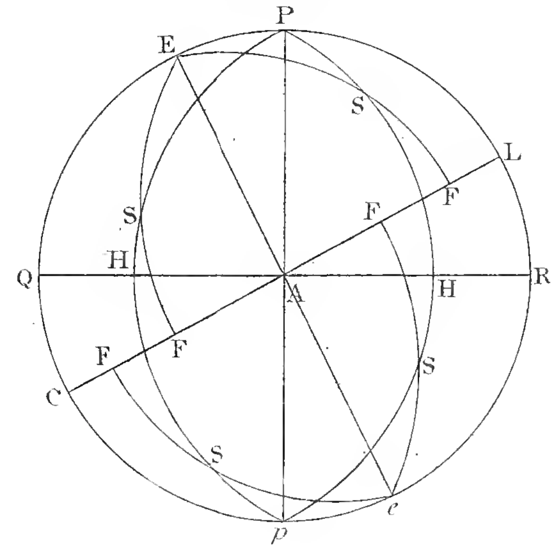


Fig. 2.

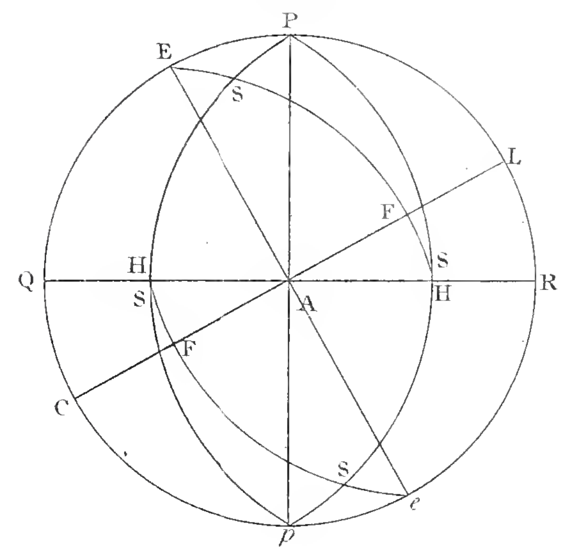


Fig. 3.

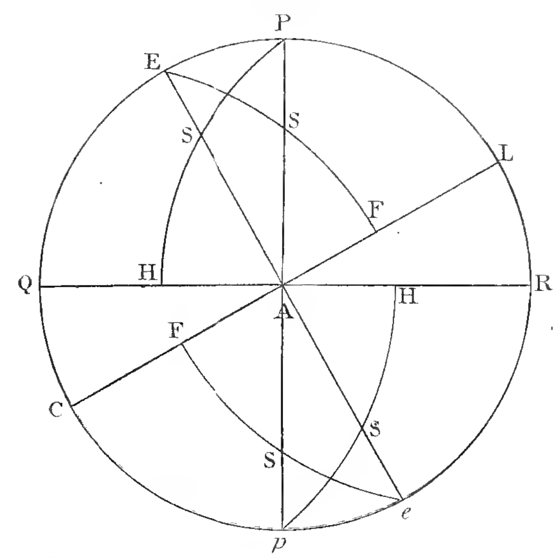


Fig. 4.

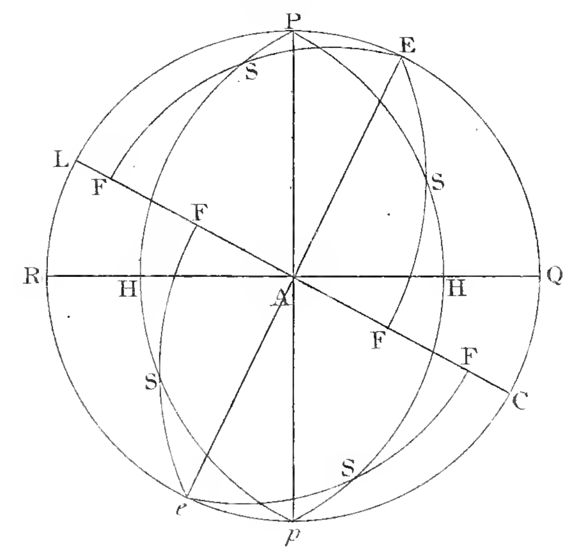


Fig. 5.

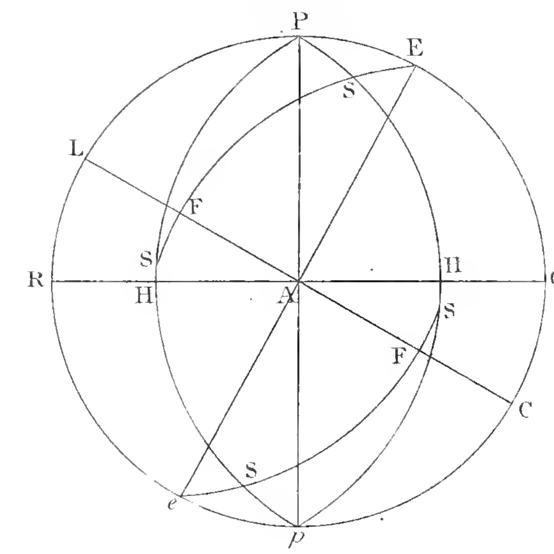
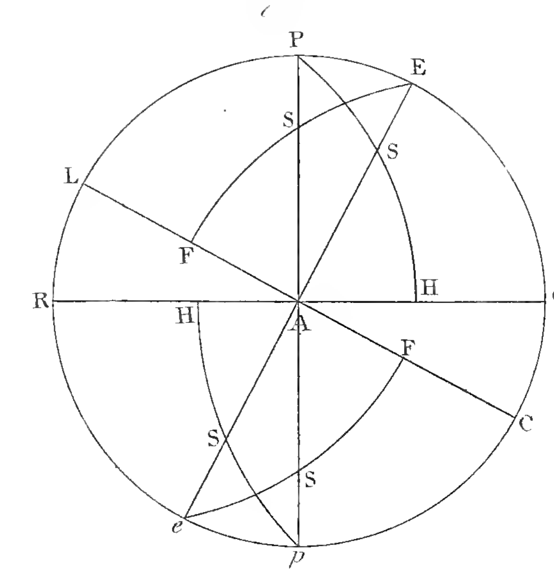


Fig. 6.



J. Basire, sc.

IX. *Some account of the feet of those animals whose progressive motion can be carried on in opposition to gravity. By Sir* Everard Home, *Bart. V.P.R.S.*

Read February 22, 1816.

THE house-fly, as is well known, is capable of walking upon the ceilings of rooms, in which situation its body is not supported on the legs; but the principle, by which it is enabled to do so, has never been satisfactorily explained, owing to the animal being too small for the feet to be submitted to anatomical investigation.

I was not aware that any animal of a much larger size was endowed by nature with a power at all similar, so as to admit of this very curious principle being investigated, till Sir JOSEPH BANKS, a few months ago, mentioned that the Lacerta Gecko, a native of the island of Java, comes out of an evening, from the roofs of the houses, and walks down the smooth hard polished chinam walls, in search of the flies that settle upon them, which are its natural food, and then runs up again to the roof of the house. Sir JOSEPH BANKS, while at Batavia, amused himself in catching the Lacerta Gecko, by standing close to the wall, at some distance from the animal, with a long flattened pole, which being made suddenly to scrape the surface of the wall, knocked the animal down.

He has procured for me a specimen of a very large size,

weighing five ounces three quarters, avoirdupoise weight, which has enabled me to ascertain the peculiar mechanism by which the feet of this animal can keep their hold of a smooth hard perpendicular wall, and carry up so large a weight as that of its body.

The foot of the Gecko has five toes, at the end of each of which, except the thumb, is a very sharp claw much curved; on the under surface of each toe are sixteen transverse slits, leading to so many cavities or pockets, the depth of which is nearly equal to the length of the slit that forms the orifice; they all open forwards, and the external edge of each opening is serrated, like the teeth of a small-toothed comb. The cavities, or pockets, are lined with a cuticle, and the serrated edges are covered with it. On each side of the bones of the toe, which are three in number, is situated a large muscle of an oval form; its origin is at the tarsus, the fleshy portion extends to the end of the first bone of the toe, and the tendons of both are continued on to the claw, which is moved by these muscles. From the tendons of these large muscles, two sets of smaller muscles originate; one pair of which is lost upon the posterior surface of each of the cavities, or pockets, that lie immediately over them.

The large muscles, by their contraction, draw down the claw, and necessarily put the small muscles that go off from the tendons of the larger upon the stretch, so that under such circumstances they act to a greater advantage. When these contract, they open the orifices of the cavities, or pockets, to which they belong, and turn down the serrated edge upon the surface on which the animal stands.

On each side of the toes there is a loose fold of skin, giving

the toes an unusual breadth. The cavities, or pockets, which have been described, and the muscles connected to them, form the only peculiarities in the foot of this particular species of lizard.

Upon examining attentively the under surfaces of the toes, when the cavities, or pockets, are closed, they bear a considerable resemblance to the surface of that portion of the head of the Echineis Remora, or sucking-fish, by which it attaches itself to the shark, or the bottom of ships; it therefore suggested itself, that much useful information, applicable to the present subject, might be derived from the examination of such an apparatus, more especially as the parts of which it is composed, are so much larger in size, and more within the the reach of examination.

The surface on the top of the head of the Echineis Remora, fitted for adhesion, is of an oval form, and bears a considerable proportion to the size of the whole animal; it is surrounded by a broad, loose, moveable edge, capable of applying itself closely to the surface on which it is placed.

The apparatus itself consists of two rows of cartilaginous plates connected by one edge to the surface on which they are placed; the other, which is external, having the same serrated appearance described in the mechanism of the toes of the Lacerta Gecko. These plates are capable of being raised and depressed at the will of the animal, there being muscles upon the skull adapted to that purpose. The two rows are separated by a thin ligamentous partition, and the only apparent reason for their being so divided, is to render them more manageable, as the two portions in every respect resemble one another.

It is evident, that when the external edge of this apparatus is closely applied to any surface, and the cartilaginous plates are raised up, the interstices must become so many vacua, and the serrated edge of each plate will keep a sufficient hold of the substance on which it rests, to retain it in that position, assisted by the pressure of the surrounding water, without a continuance of muscular exertion.

It thus appears, that the adhesion of the Echineis Remora is produced by so many vacua being formed by an apparatus worked by the voluntary muscles of the animal, and the pressure of the surrounding water.

From the similarity of the mechanism of the under surface of the toes of the Lacerta Gecko, there can be no doubt, that the purpose to which it is applied, is the same; but as in the one case, the adhesion is to take place under water, and is to continue for longer periods, the means are more simple. In the other, where the mechanism is to be employed in air, under greater disadvantages with respect to gravity, and is to last for very short periods, and then immediately afterwards be renewed, a more delicate structure of parts, a greater proportional depth of cavities, and a more complex muscular structure becomes necessary.

Having ascertained the principle on which an animal of so large a size as the Lacerta Gecko, is enabled to support itself in its progressive motion against gravity, I felt myself more competent to enquire into the mechanism by which the common fly is enabled, with so much facility, to support itself in still more disadvantageous situations.

In the natural size the feet of the fly are so small, that nothing can be determined respecting them; and when highly

magnified, such is the liability to error, that any person with a preconceived opinion becomes an improper observer of the appearances that are represented. From this consideration, I have not examined them myself, but have rather chosen to refer to the representation of their structure taken by others. Mr. GEORGE ADAMS, mathematical and optical instrument maker, in Fleet-street, London, in the year 1746, published a plate representing the appearance of the fly's foot when highly magnified. This figure will be found at the end of the paper. His account of the uses of the different parts is by no means satisfactory, but he concludes it by saying, "That the fly is enabled to walk on glass, proceeds partly from a ruggedness of the surface, or a kind of tarnish, or dirty, smoaky substance adhering to the surface of that very hard body; and though the pointed parts *(of the fly's foot)* cannot penetrate, yet they may find pores enough in the tarnish, or at least make them. This structure Mr. HOOK surveyed with great diligence, because he could not comprehend, that if there was any such glutinous matter in those supposed sponges (as most that have observed that object in a microscope have believed), how the fly could so readily unglew and loosen its feet; and also because he had found no other creature any ways like it." JEAN CHRISTOFLE KELLER, painter at Nuremberg, made a drawing of the fly's foot in a highly magnified state, which was published in 1766. The author of the publication to which these plates are annexed, whose name is not mentioned, takes some pains to refute the opinion of M. REAUMUR, who calls the surfaces of the soles of the fly's feet *pelotes*, or balls, which this author ascribes to M. REAUMUR not having seen them sufficiently distinctly. This author says, that they are not balls,

but concave surfaces, as KELLER represents them; a copy of which representation is annexed.

Although the author states them to be concave surfaces, he says that they are only used when the fly moves horizontally; but when it moves perpendicularly, or upon the ceiling, they are turned up out of the way; and the progressive motion is carried on by fixing the crotchets into the irregularities of the surface on which the fly treads, whether glass, porcelain, or any other substance. It will, however, scarcely be doubted, from the preceding facts, that these concave surfaces are employed to form vacua, which enable the fly to move under such disadvantageous circumstances upon the same principle as the Lacerta Gecko.

EXPLANATION OF PLATES.

PLATE VII.

The external form of the Lacerta Gecko.

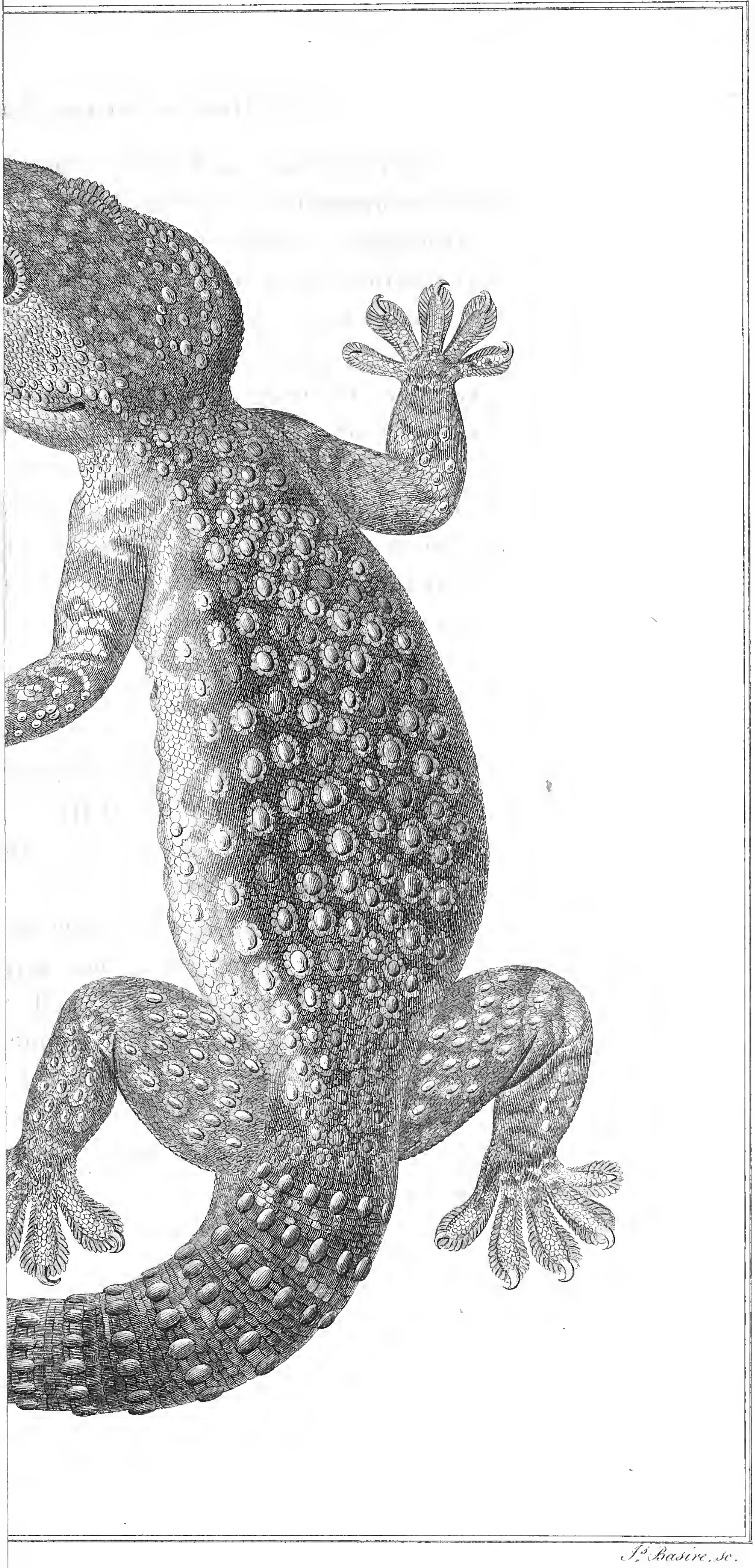
PLATE VIII.

Fig. 1. The under surface of one of the toes of the Lacerta Gecko of the natural size.

Fig. 2. A toe dissected to show the appearance of the pockets on its under surface, their serrated cuticular edge, the depth of the pockets, and the small muscles by which they are drawn open, the parts much magnified.

aa The two muscles which lie on the sides of the bones of the toe, with their tendons inserted into the last bone close to the root of the claw. From these tendons the muscles belonging to the pocket go off.

Fig. 3. The upper surface of the head of the Echineis Remora, to show the apparatus by which the animal has a



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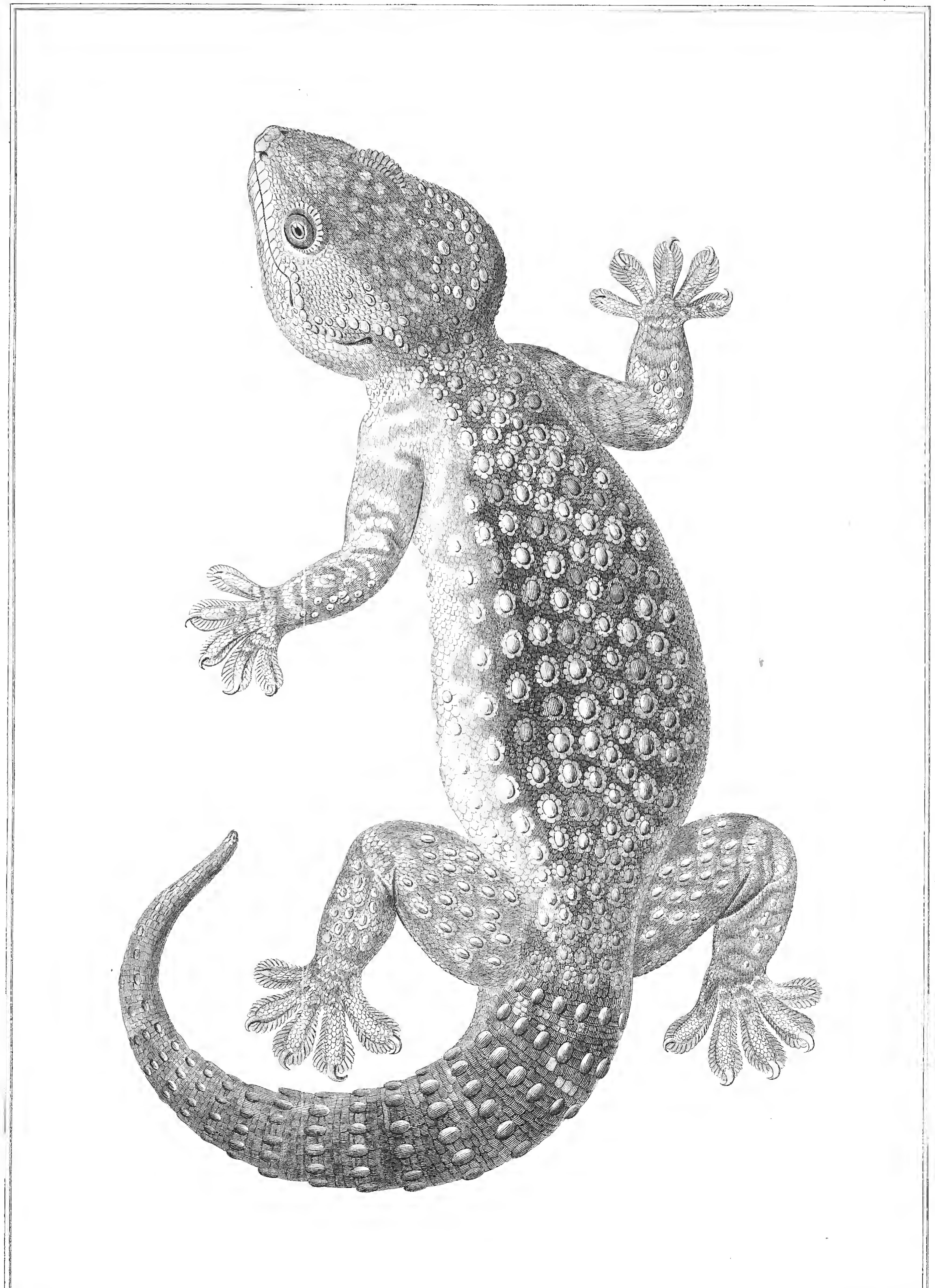

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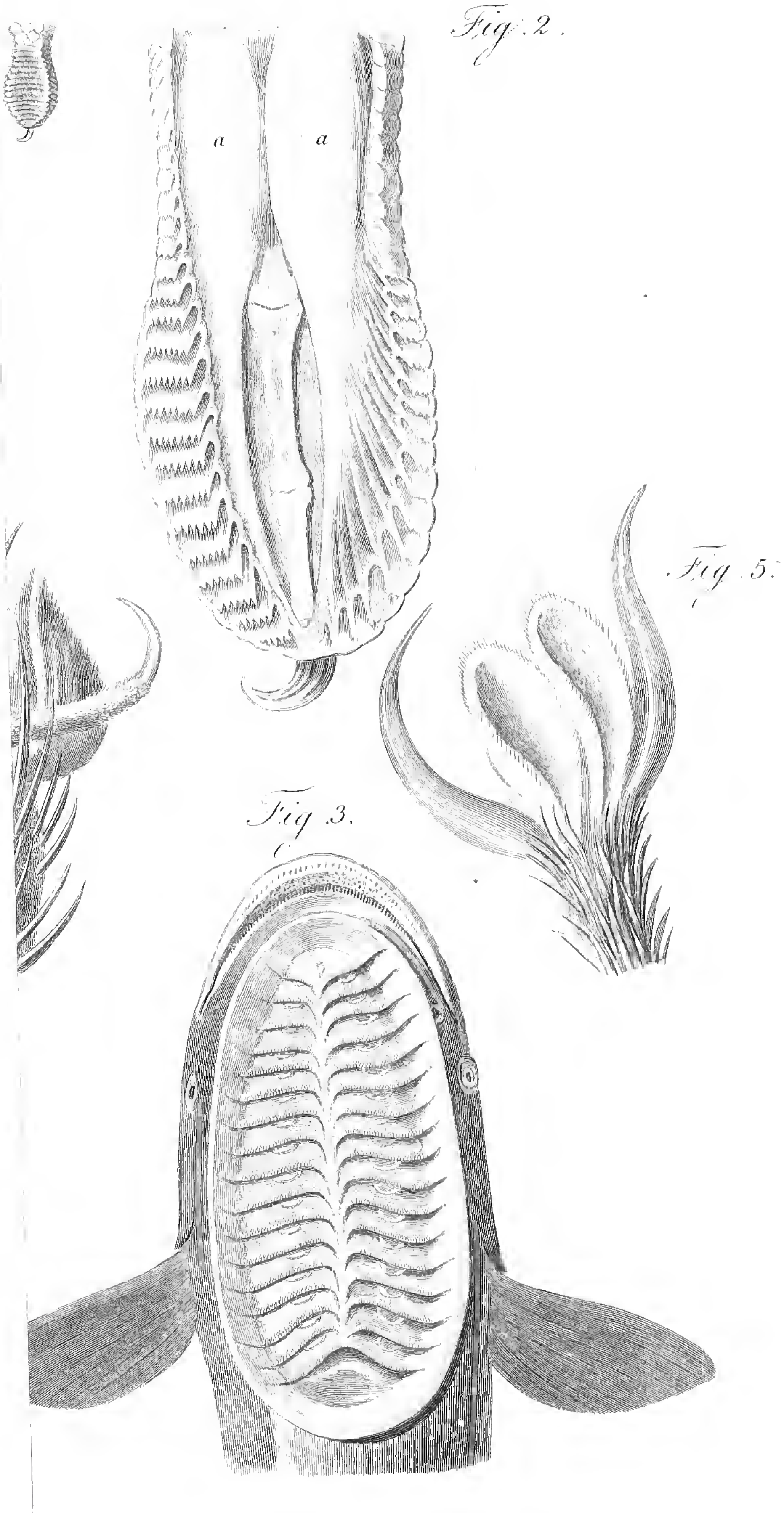

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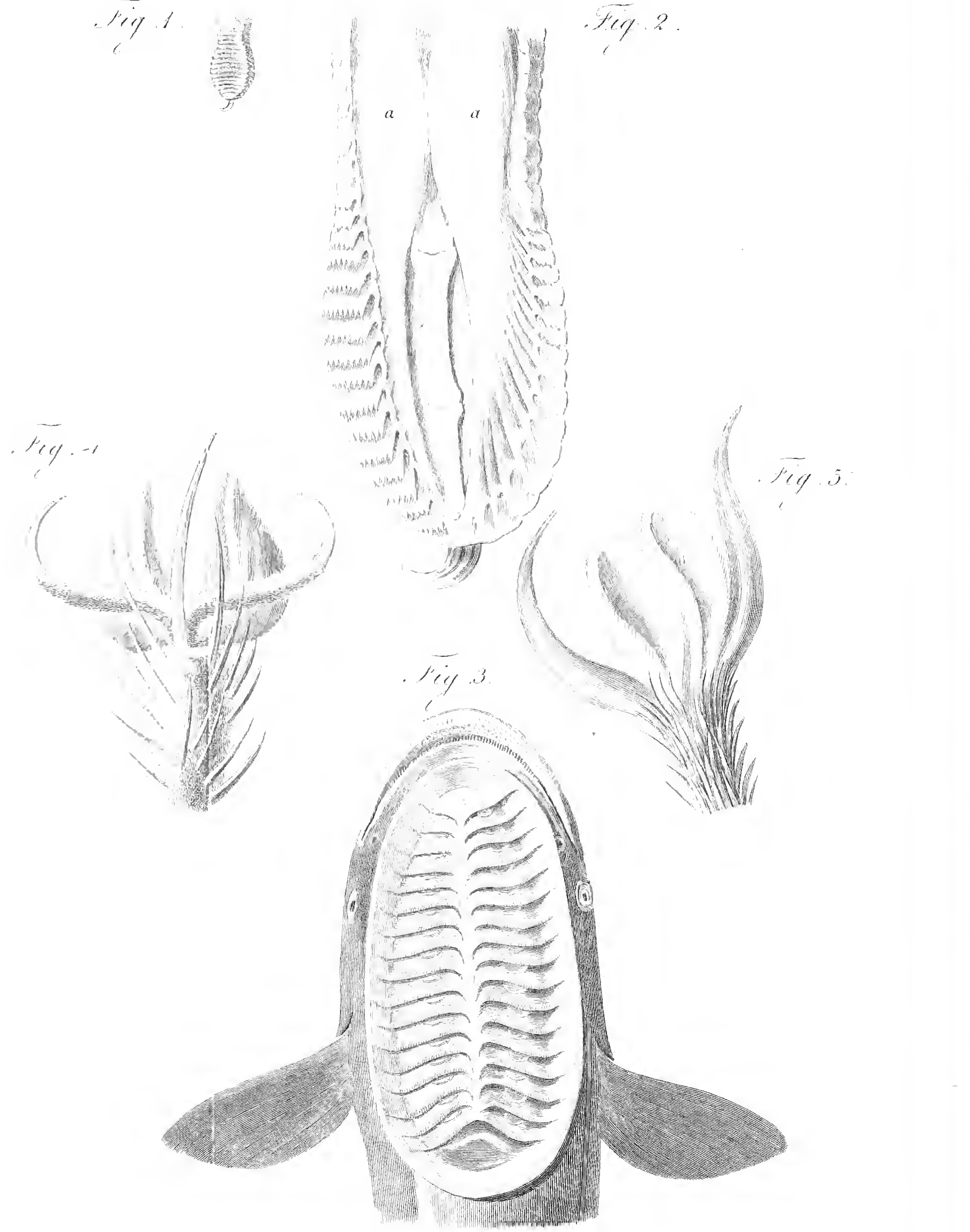

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power of adhering to the surface on which it is placed, of the natural size. One half of this apparatus has the cartilaginous plates closed, the other open.

Fig. 4. The under surface of the foot of the house-fly highly magnified, showing the two concavities by which the foot attaches itself to the surface on which it is placed, and two claws for laying hold. Copied from a plate by G. ADAMS, published in 1746.

Fig. 5. Another view of the same parts copied from a plate published in 1766, taken from a drawing of CHRISTOFLE KELLER, painter at Nuremberg.

X. *On the communication of the structure of doubly refracting crystals to glass, muriate of soda, fluor spar, and other substances, by mechanical compression and dilatation. By* David Brewster, *LL. D. F. R. S. Lond. and Edin. In a letter addressed to the Right Hon. Sir* Joseph Banks, *Bart, G.C.B. P.R.S.*

Read February 29, 1816.

Dear Sir,

Notwithstanding the numerous discoveries which have lately been made relative to the polarisation of light, and the optical phenomena of crystallized bodies, not a single step has yet been made towards the solution of the great problem of double refraction. What is the mechanical condition of crystals that form two images and polarise them in different planes; and what are the mechanical changes which must be induced on uncrystallized bodies in order to communicate to them these remarkable properties, are questions which are as difficult to be answered at the present moment, as they were in the days of Huyghens and Newton.

In the frequent attempts which I have made to obtain a solution of these difficulties, the polarisation of light by oblique refraction was the only phenomenon that seemed to connect itself with the inquiry; but the hopes of success which this fact inspired, were soon found to be delusive, and the subject resumed its former impregnable aspect. A new train of experiments, however, has enabled me not only to give a

satisfactory answer to the questions which have been stated, but to communicate to glass, and many other substances, by the mere pressure of the hand, all the properties of the different classes of doubly refracting crystals. The method of producing these effects, and the consequences to which it leads, will be briefly explained in the following letter.

SECT. I. *On the communication of double refraction to glass, muriate of soda, and other hard solids.*

Proposition I.

If the edges of a plate of glass, which has no action upon polarised light, are pressed together or dilated by any kind of force, it will exhibit distinct neutral and depolarising axes like all doubly refracting crystals, and will separate polarised light into its complementary colours. The neutral axes are parallel and perpendicular to the direction in which the force is applied, and the depolarising axes are inclined to these at angles of 45°.

I took a plate of glass about 1 inch broad, $2\frac{1}{2}$ inches long, and 0.28 of an inch thick, and having compressed its edges by the force of screws, I found that it polarised a white of the first order in every part of its breadth. The depolarising axes formed an angle of 45° with the edges of the plate. By increasing the compressing force, it polarised a faint yellow light of the first order, which gradually rose into orange.

When the screw pressed upon the glass only at a single point, an appearance was exhibited similar to that shown in Fig. 1, (Pl. IX.) where AB is a cubical piece of glass pressed in the clamp CDE by means of the screw S. Between the points of pressure *m*, *n*, fringes *mon*, *mpn*, are developed. Between

A and *o* the tint is a white of the first order, passing into yellow at *o*, then advancing up the scale to *r*, and descending by similar gradations to B. The effect produced by turning the glass round 45° is shown in Fig. 2.

If the axis of pressure *mn*, Fig. 3, (Pl. IX.) is near one side of a plate of glass AB, an effect is produced at B exactly like the four sets of fringes exhibited by crystallized glass. When the axis of pressure *mn* is in the middle of a plate AB, Fig. 4, (Pl. IX.) about 1½ inch long, the same effect is produced towards A and B, as if the two pieces *Amn*, *Bmn*, had been crystallized separately by heat.*

I experienced considerable difficulty in applying a dilating force to glass, till I discovered the method described under Proposition III.

Proposition II.

When a plate of glass is under the influence of a compressing force its scructure is the same as that of one class of doubly refracting crystals, including calcareous spar, beryl, &c.; but when it is under the influence of a dilating force, its structure is the same as that of the other class of doubly refracting crystals, including sulphate of lime, quartz, &c.

When a plate of dilated glass was combined with a similar plate of compressed glass, so that the direction of the dilating force coincided with the direction of the compressing force, the difference of their effects was produced, and *vice versâ*. The truth in the Proposition was also established by combining the glass with standard plates of sulphate of lime.

* See Phil. Trans. 1816. p. 90, fig. 35.

Proposition III.

If a long plate or slip of glass is bent by the force of the hand, it exhibits at the same time, the two opposite structures described in the preceding Proposition. The convex, or dilated side of the plate affords one set of coloured fringes, similar to those produced by one class of doubly refracting crystals; and the concave or compressed side, exhibits another set of fringes similar to those produced by the other class. These two sets of fringes are separated by a deep black line where there is neither compression nor dilatation.

This curious result may be obtained by plates of glass of any size, provided they are a few inches in length, but the experiment is more easily made with a long and narrow slip. When a very small degree of force is employed in bending it, a faint bluish white fringe appears at both edges. As the force increases, these fringes encroach upon the interjacent black space, and gradually become *white, yellow, orange, purple, indigo, blue, green, yellow,* &c. till three or four orders of colours are distinctly developed on each side of the black space. These phenomena are represented in Figs. 5 and 6. (Pl. IX.) Fig. 5, shows the effect produced by a very small force, and Fig. 6, the effect produced by a considerable force. In one of these experiments, when the plate of glass was $1\frac{1}{2}$ inch broad, 0.28 thick, and 6 inches between the points of support, I developed by the force of a screw no fewer than 7 orders of colours. The black fringe was scarcely perceptible, and the white tint arising from the mixture of all the colours, was on the eve of being produced when the plate broke in pieces.

Scholium.

The experiments now described, furnish us with a method of rendering visible, and even of measuring the mechanical changes which take place during the compression, dilatation, or bending of transparent bodies. The tints produced by polarised light are correct measures of the compressing and dilating forces, and by employing transparent gums, of different elasticities, we may ascertain the changes which take place in bodies, before they are either broken or crushed. The subject, therefore, of the strength of materials, and the cohesion of solids, will derive new lights from the principles already established.

There is one practical application of these views which is particularly deserving of notice. In order to observe the manner in which stone arches yield to a superincumbent pressure, Dr. Robison executed several models in chalk, and deduced many general laws relative to the internal forces by which they were crushed. If the arch stones of models are made of glass, or any other simply refracting substance, such as gum copal, &c. the intensity and direction of all the forces which are excited by a superincumbent load in different parts of the arch, will be rendered visible by exposing the model to polarised light. If different degrees of roughness are given to the touching surfaces of the glass *voussoirs*, the results may be observed for any degree of friction at the joints. The intensity and direction of the compressing and dilating forces which are excited in loaded framings of carpentry, may be rendered visible in a similar manner.

Proposition IV.

The tints polarised by plates of glass in a state of compression or dilatation, ascend in Newton's *scale of colours as the forces are increased; and in the same plate, the tint polarised at any particular part is proportional to the compression or dilatation to which that part is exposed.*

We have already seen, in illustrating the preceding Propositions, that higher tints are developed as the forces are increased. If ABCD, Fig. 7, (Pl. IX.) is a plate of glass, rendered concave by bending, and *mn* the black space which separates the dilated portion AB from the compressed portion CD, then if *ef* be the natural distance of the particles of the glass, and *ab* their distance when dilated at the convex edge AB, *cd* will represent the distance of two particles situated at *c*, and the tint at *c* will be to the tint at *a*, as *cd—ef* is to *ab—ef*; but *cd—ef* : *ab—ef* : : *ec* : *ea*; and therefore the tints at any part *c* will be proportional to its distance *ce* from the limit of compression and dilatation. The fringes developed on each side of *mn* have nearly the same breadth, which clearly shows that the tints are proportional to the actual compressions and dilatations.

Proposition V.

When compressed and dilated plates of glass are combined transversely and symmetrically, they exhibit all the phenomena which are produced by the combination of plates of doubly refracting crystals.

If a plate of compressed glass is combined *symmetrically* with a similar plate, the tint polarised by the combination is

that which is due to the sum of their thicknesses; but if they are combined *transversely*, the effect is that which is due to the difference of their thicknesses. The same is true of plates of dilated glass.

If a plate of compressed glass is combined symmetrically with a plate of dilated glass, the effect is that which is due to the difference of their thicknesses; while a transverse combination gives an effect due to the sum of their thicknesses. The action of plates of compressed and dilated glass are regulated by the same laws which M. BIOT has investigated for the different classes of doubly refracting crystals.

In order to observe the effects of crossing plates of glass that possess both structures, I took a stiff bar of iron AB, Fig. 8., (Pl. IX.) and placed upon it the glass plate CD, which was separated from the iron by the supports E, F; and by means of the screw S, I kept it in such a bent state, that it exhibited the fringes shown in Fig. 5. (Pl. IX.) When this plate was crossed by another similar plate at right angles, the intersectional figure had the form shown at Fig. 9. (Pl. IX.) At the angles *o, p*, where the dilated portions cross the compressed portions, the colours rise in the scale, and the maximum tints of each plate are exactly doubled at the angular point; but at the other angle *m, n*, where the dilated portion of the one, crosses the dilated portion of the other, or where the compressed portions cross each other, the tints of the one plate are counteracted by those of the other, and therefore a black fringe *mn*, extends across the diagonal of the intersectional figure.

When a plate of bent glass is crossed by a plate of glass crystallized by heat, as shown in Fig. 10, (Pl. IX.) it produces an intersectional figure which can easily be determined *a priori*,

and which is exactly one half of the intersectional figure that would be produced by crossing AB with another plate of crystallized glass, having the same tints as CD in its four sets of fringes.

Proposition VI.

If a plate of glass resting on two supports, is bent by any force applied between the points of support, the tints are a maximum at the part where the pressure is applied, and ascend gradually in the scale of colours towards the points of support.

I took a plate of glass ABCD, Fig. 11, (Pl. IX.) six inches long, $1\frac{1}{2}$ broad, and 0.28 thick, and having placed its extremities upon the points of support C, D, I bent it by a screw applied to the surface at M. Seven orders of colours were now distinctly seen on each side of the black fringe in the section M*m*. In the sections 1.1, the first order of colours only was developed; between the sections 1 1, and 2 2, the second order of colours appeared, and so on with the succeeding orders, till the seventh was seen near M*m*.

Scholium.

It follows from the preceding experiments, that the mechanical contractions and dilatations at the points, 1, 2, 3, 4, 5, 6, are as the numbers, $7\frac{1}{5}$, $13\frac{11}{20}$, 22, $29\frac{2}{3}$, 38, $45\frac{4}{5}$, the values of the corresponding tints in Newton's scale.*

* See Newton's Optics, Book II. Part II. p. 206.

Proposition VII.

If a plate of glass is subject to compressions or dilatations exerted in different directions, the same effects are produced as when separate plates influenced by the same forces are combined in a similar manner.

I took a plate of glass AB, Fig. 12, (Pl. IX.) and having compressed its extremity A by means of the screw S, a bright white of the first order emerged from the points of pressure P,Q. By a force applied at B, I now bent the glass so as to make the lower side concave, and to produce the white tints on each side of the interjacent black space *m*, *n*. The effects of bending were now combined towards *m*, with the effects of compression, so that in the line *mo*, a black fringe appeared, the compressed structure produced by bending having acted in opposition to the compressed structure produced by the screw S; while in the line *mp*, a yellow tint emerged, the dilated structure produced by *bending*, acting in conjunction with the compressed structure produced by the screw. These results will appear perfectly conformable to Prop. V., when we consider that the axis of compression produced by the screw is PQ, while the axis of compression and dilatation produced by bending is parallel to *m n*, and consequently at right angles to PQ.

PROPOSITION VIII.

If two plates of bent glass are placed together at their concave or compressed edges, the compound plate has exactly the same properties as a plate of glass transiently or permanently crystallized by heat, which gives the usual *series of fringes. But if the two plates are placed together at their convex or dilated edges, the compound plate has the same properties as plates of glass transiently crystallized by heat, which produce the* unusual *series of fringes.*

The plates described in the Proposition exhibit the same intersectional figures as the plates of crystallized glass, and have in every respect the same action upon polarised light.

PROPOSITION IX.

If the compressing and dilating forces are applied to the centre of a plate of glass, the principal axes of the particles will be directed to the point of compression or dilatation, and the glass will exhibit the black cross, and the other phenomena which are seen in doubly refracting crystals.

Having procured a strong lens of considerable convexity, I pressed it by means of a screw upon the centre of a plate of glass. When exposed to polarised light, it exhibited the appearance shown in Fig. 13, (Pl. IX.) where ABCD is part of the lens, and m, n, o, p, four rectangular sectors, separated by a black cross. When the pressure is increased, different tints and fringes are developed, as in crystallized bodies.

Proposition X.

If a plate of glass in a state of compression or dilatation is inclined to the polarised ray in a plane parallel to the axis of dilatation and compression, the tints will descend in the scale; but if they are inclined in a plane at right angles to these axes, the tints will ascend.

This result was obtained by the inclination of plates compressed by screws, and of plates compressed and dilated by bending.

Proposition XI.

If a plate of glass that has already received the doubly refracting structure from heat, is exposed to compression, the tints of the interior fringes rise in the scale, and those of the exterior fringes descend, when the axis of pressure is perpendicular to the direction of the fringes; the opposite effect being produced by a dilating force. The same results are in this case obtained as if an uncrystallized plate similarly compressed or dilated, had been similarly combined with the crystallized plate.

I took a plate of crystallized glass, which displayed in the middle fringes a blue of the second order, and having compressed it by a screw in a direction perpendicular to the fringes, the tint of the interior set rose to the red of the second order, while that of the exterior set descended in the scale. When the plate was pressed in a direction perpendicular to the fringes, the tint of the interior set descended to a faint yellow, and that of the exterior set rose in a similar proportion.

When a piece of crystallized glass is bent by a screw, as in Fig. 8, (Pl. IX.) the exterior fringes on the upper or concave side

increase in number and encroach on the interior fringes; but on the lower or convex side, the fringes diminish in number, and are encroached upon by the interior fringes. Uncrystallized plates, when compressed or dilated, exhibit similar effects if combined with crystallized plates not subjected to compression or dilatation.

Proposition XII.

Muriate of soda, fluor spar, diamond, obsidian, semi-opal, horn, tortoise-shell, amber, gum copal, caoutchouc, rosin, phosphorus, the indurated ligament of the chama gigantea, and other substances, that have not the property of double refraction, or that have it in an imperfect manner, are capable of receiving it by compression or dilatation.*

Of all the substances mentioned in the Proposition, obsidian, muriate of soda, and gum copal, receive from pressure the greatest polarising force. Gum copal, in particular, exhibited a greater number of fringes than a piece of glass subjected to the same pressure.

Proposition XIII.

Calcareous spar, rock crystal, topaz, beryl, and other minerals that already possess in a high degree the doubly refracting structure, suffer no change by compression or dilatation.

The state of compression or dilatation in which the particles of these crystals are already placed, according to the class in which they belong, is so great, as not to experience any change from the application of ordinary forces. I have

* See the *Edinburgh Transactions*, Vol. VIII. Part I. where I have shown that diamond, muriate of soda, &c. possess, imperfectly, the structure of both classes of doubly refracting crystals.

applied in the direction both of their neutral and depolarising axes, forces so great as to break the shoulders of all the clamps that were employed.

Proposition XIV.

To construct a chromatic dynamometer for measuring the intensity of forces.

In almost every dynamometer, which has hitherto been constructed, it is assumed that a steel spring recovers its original shape after repeated bendings, and upon this assumption the scale of the instrument is formed.* The perfect elasticity of glass, however, renders it, in this respect, a much fitter substance than steel, and though it does not admit of such a great change of shape, yet the slightest variations in its structure can be rendered visible.

If a number of narrow and thick plates of glass AB, Fig. 14, (Pl. IX.) are firmly fixed at each end in brass caps A, B; then if any force is applied to a ring at C in the middle of the plates, when the ends A and B are fixed, or if C is fixed, and the force applied at the points A, B, the plates of glass will be bent in the middle, and the force by which this is produced, will be measured by the tints that appear on each side of the black space *mn*. By diminishing the length of the plates, or increasing their number, they may be made to resist and to measure any degree of force. When the force to be ascertained is small, a single plate of glass will enable us to measure its intensity with great exactness.

* In the article Dynamometer, in the Edinburgh Encyclopedia, Vol. VIII. 299, I have described an instrument in which a variable measure of force is obtained by raising a metallic cylinder out of a fluid.

PROPOSITION XV.

If a parallelopiped of glass is enclosed on all sides, except two, in a mass of fluid metal, the contractions and dilatations which the metal experiences in passing to a state of permanent solidity, will be rendered visible by the communication of the doubly refracting structure to the glass.

I took a cylinder of tin plate AB, Fig. 15, (Pl. X.) open at both ends, and having placed a piece of glass CDEF on its lower edge EF, I surrounded it with melted lead. As soon as the lead lost its fluidity I exposed it to a polarised ray, and found that the glass exhibited no colour. As the metal contracted in its dimensions, there appeared a bluish white tint, which gradually rose through all the tints of the first order, and reached the red of the second order, when plunged in a freezing mixture.

The same result was obtained when the glass was surrounded by tin; but when it was incased in the fusible metal, consisting of eleven parts of bismuth, three of lead, and five of tin, it exhibited after cooling the same tints as if it had been dilated. In order to examine this point with greater care, I exposed the glass to a polarised ray as soon as the fusible metal was fixed. It then displayed no tints whatever, but as the cooling advanced, a tint appeared which rose to a yellow of the first order, as if the glass were highly compressed. At a certain temperature, however, the tints gradually diminished, and passed into the opposite tints produced by dilatation. Hence it follows, that after the fusible metal has assumed the solid state, it contracts its dimensions, and at a certain temperature is again expanded.

When the fusible metal assumed a settled state, I was surprised to observe, that the tint over its surface CD, Fig. 16, (Pl. X.) was not uniform, but had a curved black space *mno*, which inclosed a faint tint belonging to a compressed structure, while the other part had a faint yellow tint belonging to a dilated structure. This appearance arose from the piece of glass CD not being placed in the middle of the tin cylinder as shown in the figure. The distance E*e* was 0.74 of an inch, while F*f* was 0.97, and as the dilating force was greater in the direction *f*F than in the opposite direction *e*E, and the resistances unequal, a slight concavity would take place at *e*, and produce the black space, and the two opposite structures.

Scholium.

The results contained in this Proposition lead to the construction of new instruments for measuring the contraction and dilatation of all substances whatever, whether they are produced by variations in their temperature, or in their humidity. Hence we obtain measures also of the degrees of temperature and humidity by which these mechanical changes are produced.

A plate of glass inclosed in metal, as shown in Fig. 15, (Pl. X.) forms a *chromatic thermometer* different from the one I have described in a former paper.* In the present instrument, the tints are produced by the difference of pressures upon the glass, occasioned by the difference of expansions arising from changes of temperature; whereas, in the other instrument, the tints originate immediately from the changes of temperature. The exterior case of the thermometer repre-

* Phil. Trans. 1816, p. 108.

sented by AB in Fig. 17, (Pl. X.) may even be made of iron, brass, or any other metal that is not easily fused. And when this ring is brought to a high degree of heat, fluid lead, or tin, may be poured into the centre of it, so as to be immediately in contact with the piece of glass CD.

A *chromatic hygrometer* may be constructed by surrounding a piece of glass with a mass of any hygrometric substance, that readily absorbs moisture. This substance may be advantageously inclosed in a piece of glass or earthen ware, perforated in different places to admit the air freely.

Instead of measuring the direct pressure occasioned by contraction or expansion, the magnitude of the scale would be increased by employing these forces to bend a long slip of glass, as in Fig. 18, (Pl. X.) where AB is the glass resting against fixed supports A,B, and CD a mass of lead, or a hygrometric substance, resisted by the support E F, and altering the curvature of AB, by its contractions or dilatations. If the expanding mass CD Fig. 19, (Pl. X.) is made to act on the two extremities, A,B of the glass plate fixed at the middle M, it may sometimes be concave towards C, and sometimes convex, and the limit between these two states may be taken for the zero of the scale.

SECT. II. *On the communication of double refraction either transiently or permanently to animal jellies by gradual induration, and by mechanical compression and dilatation.*

PROPOSITION I.

When a plate of animal jelly, either approaching to fluidity, or in a state of high elasticity is compressed or dilated, it possesses the same optical properties as compressed or dilated glass.

It would be unnecessary labour to detail the numerous experiments by which I obtained from animal jellies, the various results described in the preceding Section.* I shall, therefore, content myself with pointing out a very simple method by which the experiments may be easily repeated. Let a parallelopiped of isinglass EF, Fig. 20, (Pl. X.) newly coagulated, be cemented by isinglass of the same consistency to two plates of glass AB, CD. By forcing the plates together, so as to compress the jelly, various orders of colours will be developed at *m n*, having the same character as the external fringes of crystallized glass. When the pressure is removed, two black fringes meet, as it were, at *m n*, and upon separating the plates, so as to dilate the jelly, another set of fringes will appear at *m n*, having a character opposite to that of the other fringes. If we force the plates together obliquely, so as to form an angle, and thus compress the jelly on one side, and dilate it on the other, the two opposite sets of fringes will be distinctly seen.

When the plates are pressed together with such force as to

* See Phil. Trans. 1814, p. 60. where I have given an account of the discovery of this property of animal jellies.

destroy the structure of the mass, the different tints are arranged like those of the finest variegated marble, an effect exactly similar to what I have observed in numerous specimens of the diamond, and also in mixtures of rosin and white wax.

By bringing the jelly into such a state that it is capable of being bent; by coagulating it in glass troughs; by applying dilating and compressing forces to a central point; and by stretching it in thin elastic films over plates of compressed or dilated glass, a number of interesting results will be obtained.

Proposition II.

If a parallelopiped of jelly is allowed to indurate by exposure to the air, it will acquire at its edges a variable density, similar to that produced by pressure, and its edges will act upon light like doubly refracting crystals.

Having poured some melted isinglass into a glass trough, and exposed it sometime after to polarised light, I observed a narrow and faint bluish stripe of the first order, on looking through the upper stratum. After a lapse of six hours, the tint became a brilliant white of the first order, and the stratum of jelly had depolarising axes inclined 45° to its length. In order to examine the mechanical change which the stratum had undergone, I looked through it at a small circular aperture. This aperture was elliptical, and its ellipticity gradually increased as the pencil passed nearer to the surface of the indurated stratum. Hence it follows, that the depolarising structure was produced or accompanied by a difference of density.

Proposition III.

If a plate of jelly partially indurated, is kept in a state of compression or dilatation till the induration is completed, it will acquire permanently the structure of doubly refracting crystals.

I experienced considerable difficulty in endeavouring to fix a plate of jelly in a state of permanent distension. The first process which was successful, consisted of taking a plate of isinglass, and allowing its two extremities to indurate, while the intermediate part was kept moist between two plates of glass. The isinglass was suspended by one of its indurated extremities, and dilated by a weight hanging from the other. In this distended state it exhibited very brilliant fringes, and it preserved the same property when it was completely hardened.

In order to obtain more perfect specimens, and a greater variety of forms I poured fluid isinglass into troughs of different shapes made either of glass or of soft porous wood. The effect produced by transmitting polarised light perpendicularly through one of these troughs, is shown in Fig. 21. (Pl. X.); and when the plate was inclined in the direction AB, it had the appearance shown in Fig, 22. (Pl. X.) After standing another day it exhibited, at a vertical incidence, the fringes shown in Fig. 23, (Pl. X.) where A *m* D, B *m* C, are the black spaces, and E, F, G, H, the tints of the first order of colours extending to the indigo of the second order; but by inclining the plate in the direction AB, the dark spaces A *m* D, B *m* C approximated at the points *m*, *n*, till they met

and formed a black space as in Fig. 21. (Pl. X.) By continuing the inclination, the black space opened, and gradually developed the black spaces, and the colours shown in Fig. 24, (Pl. X.) the fringes between A and D, and B and C having ascended in the scale, while those between A and B and D and C had descended. At the end of other two days, three distinct orders of colours were developed; but when the isinglass had detached itself from the glass bottom of the trough, the tints again descended to the state in which they are represented in Fig. 24. (Pl. X.)

This descent of the tints will be understood from Fig. 25, 26, 27. (Pl. X.) In virtue of the capillary attraction of the sides of the trough AB, the fluid jelly rises up at the angles *a*, *b*, and being there speedily hardened from its thinness, it adheres firmly to the sides of the trough. As the process of induration advances, the plate of isinglass is gradually detached from the glass bottom, at the corners *m*, *n*, Fig. 26, (Pl. X.) but still adheres firmly at the middle *c*. Hence the isinglass is in a state of great distension between *c* and *a*, and *c* and *b*, and consequently developes several orders of colours. But when the isinglass separates from the glass bottom at *c*, which it almost always does, it takes the position shown in Fig. 27, (Pl. X.) where the distension has obviously suffered a great diminution, and consequently the tints must descend in the scale. The adhesion at *c* is sometimes so strong that the isinglass carries up a portion of the glass along with it.

The combined effect of induration and distension in a narrow glass trough is shown in Fig. 28, (Pl. X.) which represents one half of the trough. The narrow fringe produced from induration is shown at A *m n* and B *o p*, and the tints developed

in the middle are the same as those marked in the figure. When this plate was inclined in the plane AC, the following results were obtained; but from the inequalities of the plate the measures must be very rude.

	Angle of incidence from the perpendicular.
A red of the first order, - -	0°
Violet, - - -	15
Blue, - - - -	28
Green, - - - -	34
Yellow, - - - -	42
Pink red, - - -	54
Blue, - - - -	61
Green of the third order. - -	71

Proposition IV.

The polarising force of distended isinglass exceeds that of beryl, and is far greater than that of glass, whether it has received the doubly refracting structure from heat or from pressure.

A soft film of isinglass $\frac{1}{30}$ of an inch thick, developed by dilatation a blue of the second order, when it broke.

Another film $\frac{1}{50}$ of an inch thick was brought nearly to a state of induration. When it was dilated, which was done with some difficulty, it polarised distinctly the bright red of the second order.

A third film, about $\frac{1}{45}$ of an inch, and prepared after the manner described in Proposition III, polarised a red of the fifth order. By comparing this tint, which is higher than any of the rest, with a thickness of a plate of glass which gives the same tint, we shall find that the constant factor by

which we must multiply the thickness of any plate of jelly, in order to obtain the thickness of a thin plate which would afford by reflection a tint similar to its maximum tint, is $\frac{1}{654}$.

The following are the constant factors for different doubly refracting substances.

Calcareous spar, - -	$\frac{1}{19}$	according to BIOT.
Rock crystal, - -	$\frac{1}{360}$	according to BIOT.
Sulphate of lime, - -	$\frac{1}{360}$	according to BIOT.
Mica, - - -	$\frac{1}{450}$	according to BIOT.
Isinglass - -	$\frac{1}{654}$	
Beryl, - - -	$\frac{1}{720}$	according to BIOT.
Glass, - - -	$\frac{1}{12580}$	

If the isinglass were made capable of resisting a higher degree of distension it would give a constant factor, approaching still nearer to that of mica.

Upon reviewing the general principles contained in the preceding Propositions, I cannot but allow myself to hope that they will be considered as affording a direct solution of the most important part of the Problem of double refraction. The mechanical condition of both classes of doubly refracting crystals, and the method of communicating to uncrystallized bodies the optical properties of either class, have been distinctly ascertained, and the only phenomenon which remains unaccounted for, is the division of the incident light into two oppositely polarised pencils. How far this part of the subject will come within the pale of experimental inquiry, I do not presume to determine; but without wishing to damp that ardour of research which has been so happily directed towards this branch of optics, I fear that, as in the case of electrical and magnetical polarity, we must remain satisfied with refer-

ring the polarisation of the two pencils to the operation of some peculiar fluid. The new property of radiant heat which enables it to communicate double refraction to a distant part of a plate of glass, where the heat does not reside in a sensible state;—the existence of a moveable polarity in glass, whether the doubly refracting structure is communicated transiently or permanently;—and the appearance of regular cleavages varying with the direction of the axes of double refraction, are facts which render it more than probable that a peculiar fluid is the principal agent in producing all the phenomena of crystallization and double refraction.

There is one fact, however, which forms a fine connection between the aberration of the extraordinary ray and the principles established in this Paper. It has been demonstrated by an eminent English philosopher,* that every undulation must assume a spheroidal form when propagated through a minutely stratified substance, in which the density is greater in one direction than another, and I have proved by experiment that such a substance actually possesses the property of double refraction. This singular coincidence will no doubt be regarded as an argument in favour of the undulatory system.

I have the honour to be, &c.

DAVID BREWSTER.

To the Right Hon. Sir Joseph Banks, Bart. G. C. B. P. R. S. &c. &c. &c.

* See Quarterly Review, Vol II.

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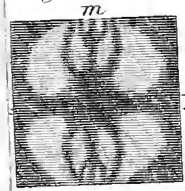
Fig. 2.



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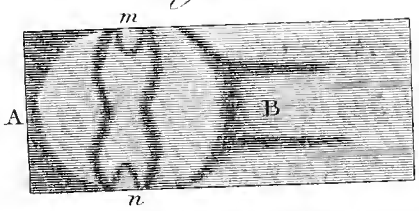

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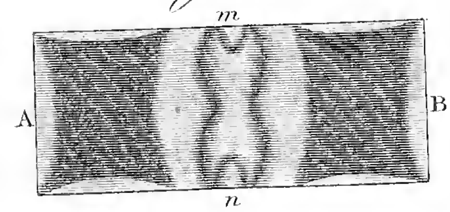

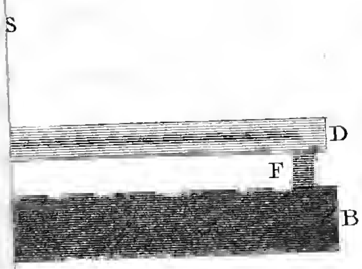

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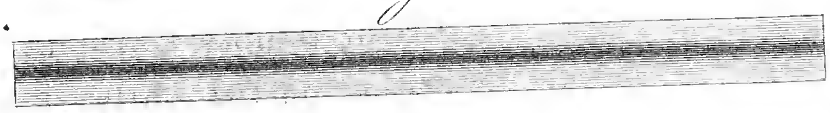

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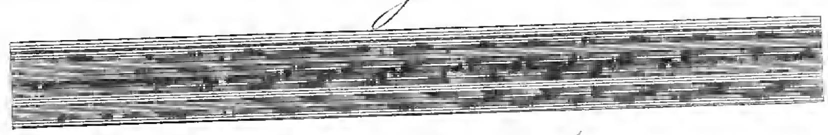

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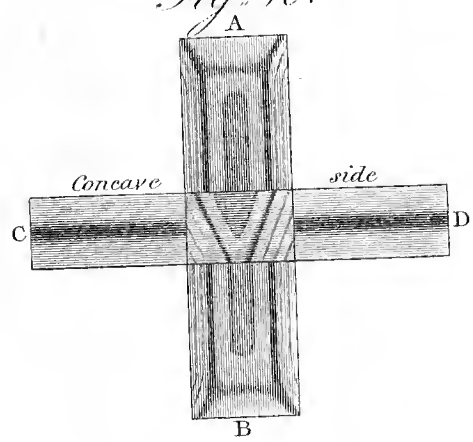

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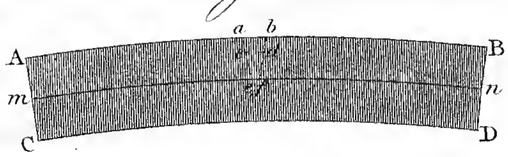

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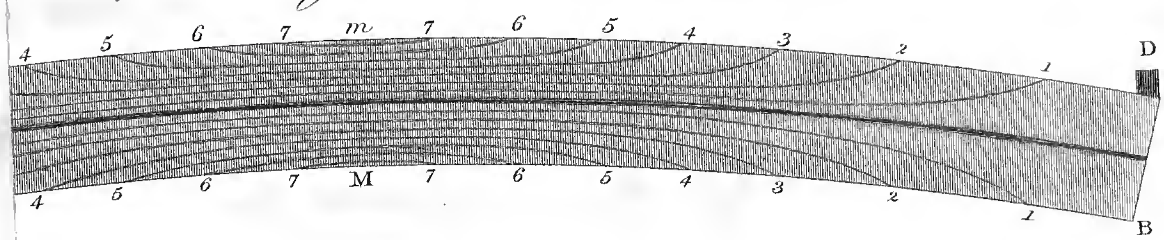

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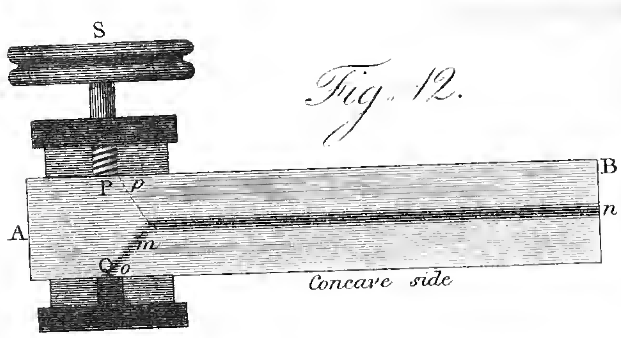

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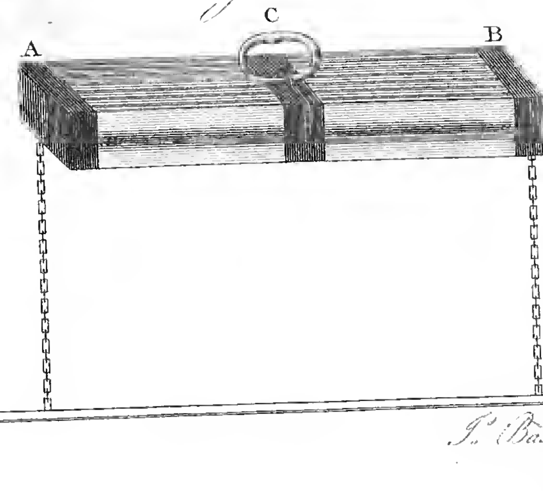

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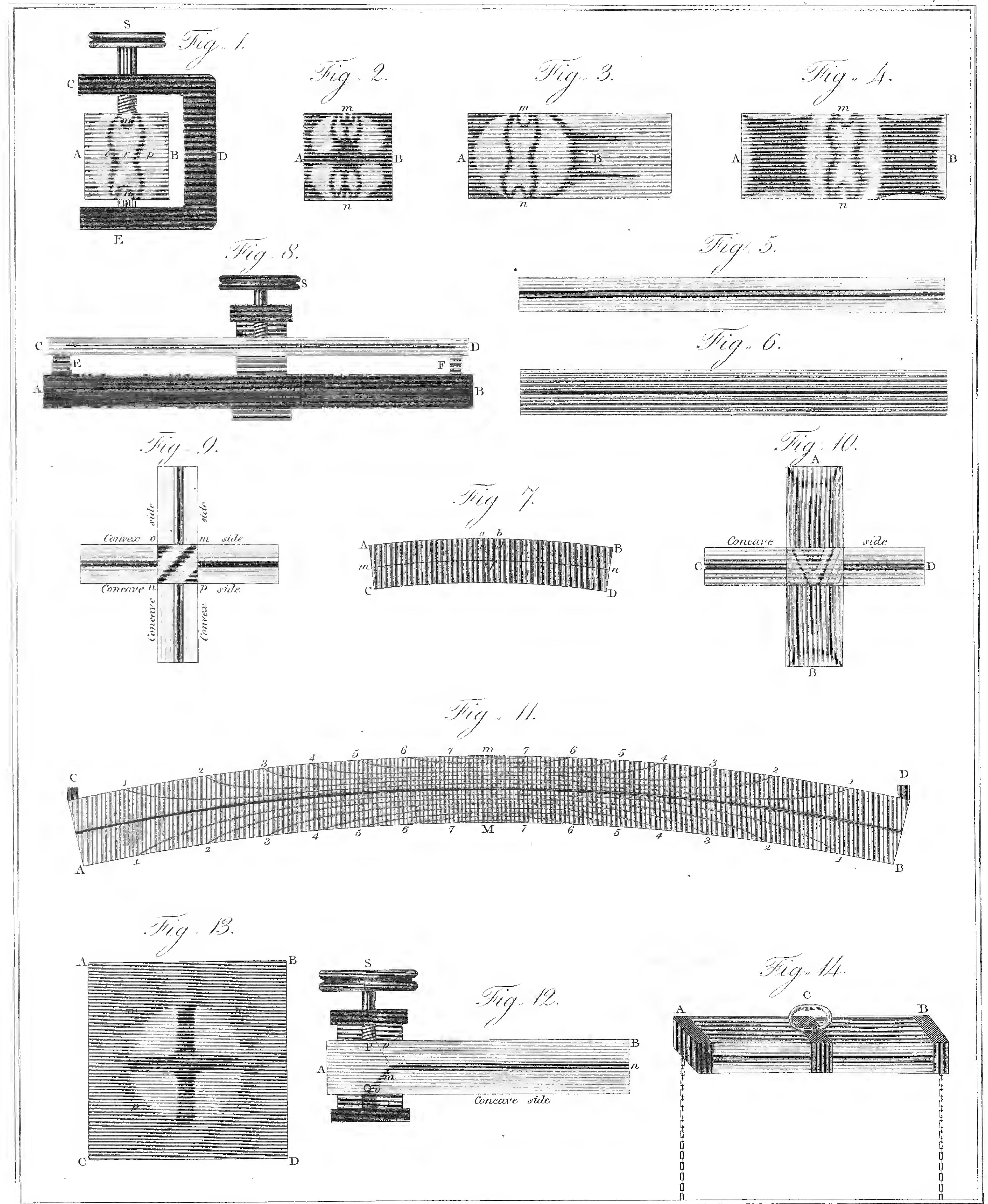
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Fig. 14.
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Concave
Convex
Concave side

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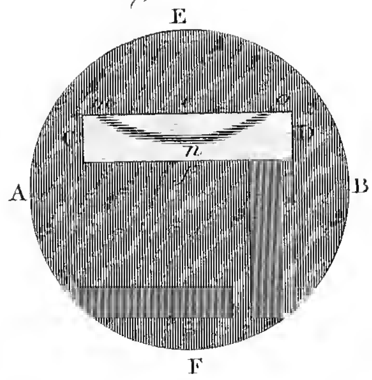
Fig. 16.



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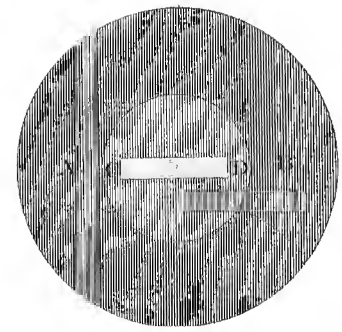

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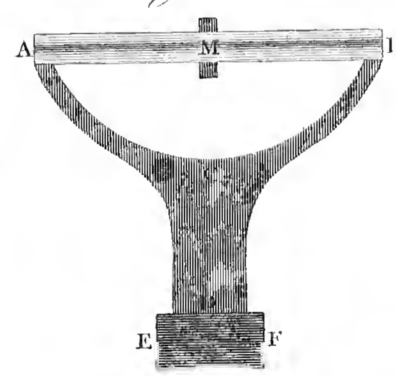

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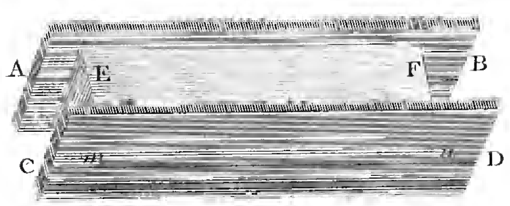

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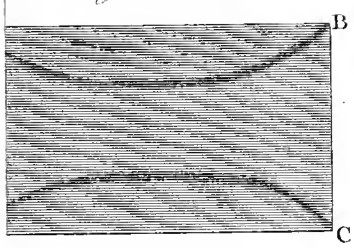

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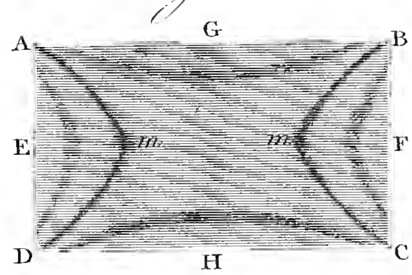

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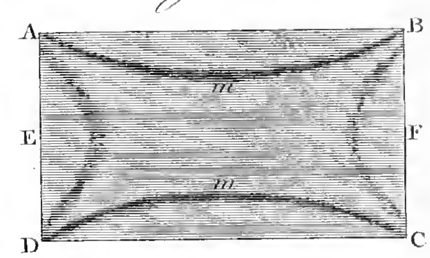

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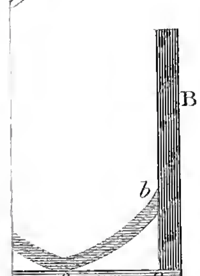

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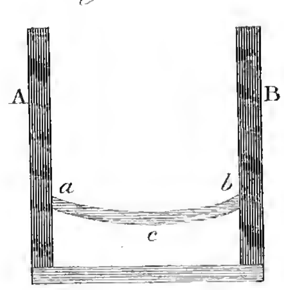
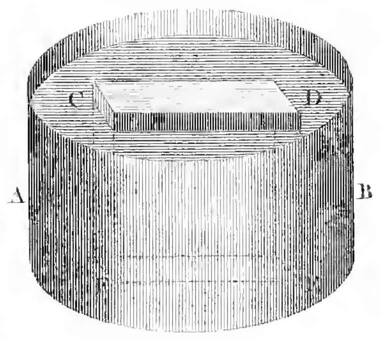


Fig. 28.

J. Basire

Fig. 15.



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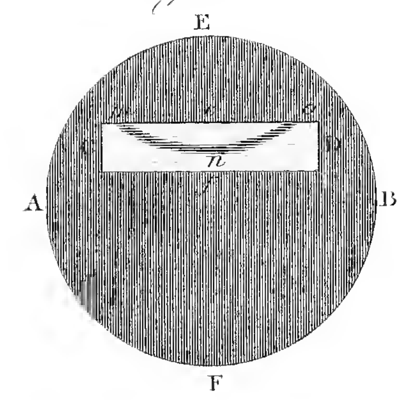

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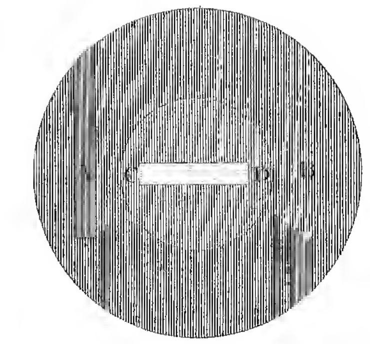

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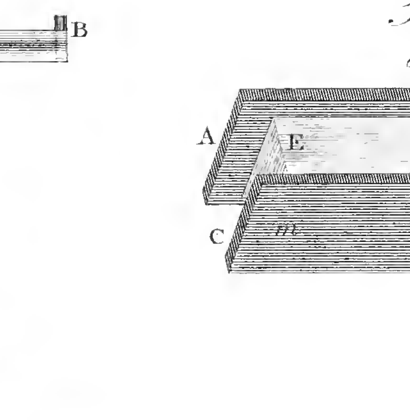

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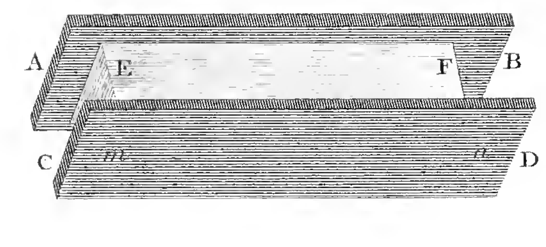

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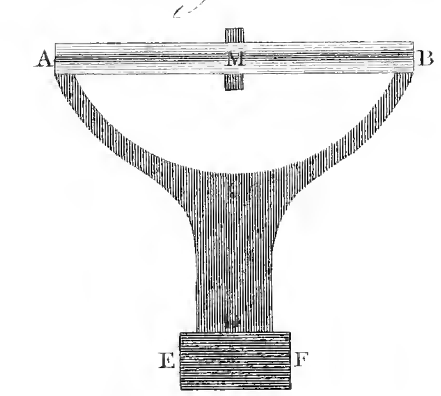

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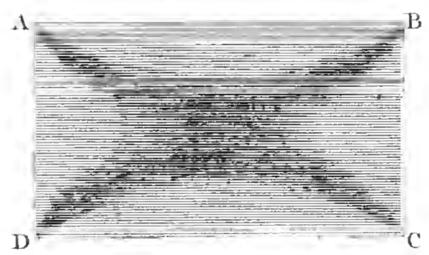

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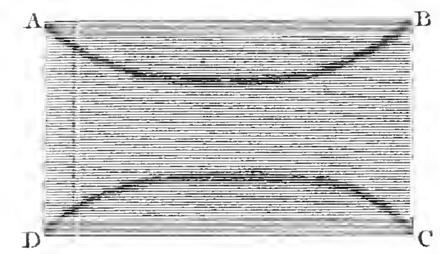

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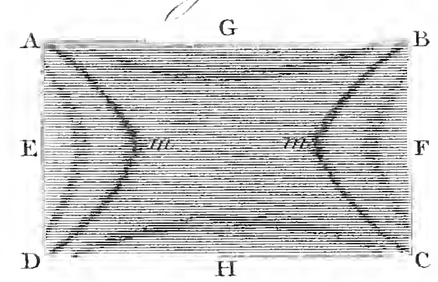

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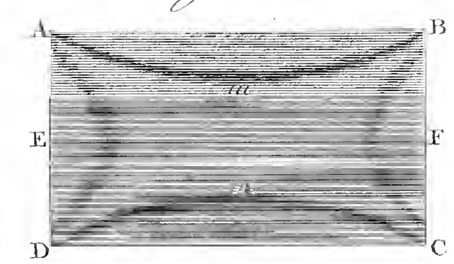

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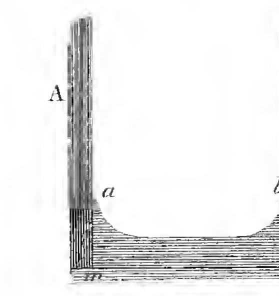

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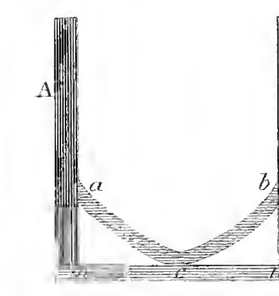

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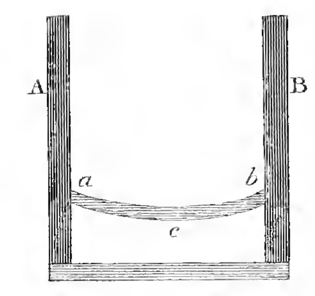

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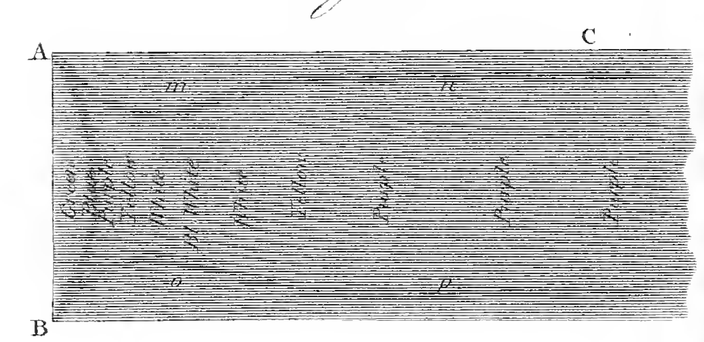

J. Basire

METEOROLOGICAL JOURNAL,

KEPT AT THE APARTMENTS

OF THE

ROYAL SOCIETY,

BY ORDER OF THE

PRESIDENT AND COUNCIL.

METEOROLOGICAL JOURNAL

for January, 1815.

1815		Time. H.	M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer.		Winds. Points.	Str.	Weather.
Jan.	1	8	0	36	47	30,20	°76	Rain this Month 0,435 Inches.	W	1	Hazy.
		3	0	41	48	30,29	72		NW	1	Fine.
	2	8	0	35	46	30,46	78		N	1	Foggy.
		3	0	36	48	30,47	77		NE	1	Cloudy.
	3	8	0	33	47	30,45	69		E	1	Cloudy.
		3	0	35	50	30,37	68		N	1	Cloudy.
	4	8	0	32	45	30,21	68		E	1	Cloudy.
		3	0	34	47	30,16	69		NE	1	Cloudy.
	5	8	0	31	43	30,10	71		N	1	Cloudy.
		3	0	34	47	30,11	71		NE	1	Fine.
	6	8	0	32	43	30,07	70		NW	1	Cloudy.
		3	0	35	47	30,07	69		E	1	Cloudy.
	7	8	0	29	43	30,01	70		E	1	Cloudy.
		3	0	34	44	29,83	71		W	1	Cloudy.
	8	8	0	33	42	29,51	75		W	1	Cloudy.
		3	0	35	43	29,57	70		NW	1	Fine.
	9	8	0	27	41	30,05	71		W	1	Hazy.
		3	0	35	46	30,02	70		W	1	Fine.
	10	8	0	41	44	29,78	78		NNW	1	Cloudy.
		3	0	41	49	29,71	69		W	1	Cloudy.
	11	8	0	38	46	29,56	68		W	1	Fine.
		3	0	39	50	29,64	62		W	1	Fine.
	12	8	0	35	45	29,99	70		NW	1	Hazy.
		3	0	38	49	30,09	68		NW	1	Fine.
	13	8	0	28	44	30,18	71		W	1	Hazy.
		3	0	37	49	30,08	71		S	1	Rain.
	14	8	0	38	46	29,82	70		N	1	Rain.
		3	0	43	50	29,82	70		N	1	Cloudy.
	15	8	0	34	47	30,19	70		SE	1	Hazy.
		3	0	34	45	30,27	66		E	1	Cloudy.
	16	8	0	31	43	30,33	68		N	1	Hazy.
		3	0	34	47	30,27	68		W	1	Cloudy.

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1815	Time. H.	Time. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer.		Winds. Points.	Winds. Str.	Weather.
Jan. 17	8	0	36	45	30,28	74	Rain this Month 0,435 Inches.	N	1	Cloudy.
	3	0	38	50	30,31	68		N	1	Cloudy.
18	8	0	32	44	30,18	67		N	1	Fine.
	3	0	35	47	30,13	70		N	1	Snow.
19	8	0	29	43	30,02	73		N	1	Cloudy.
	3	0	32	48	29,90	70		N	1	Fine.
20	8	0	30	42	29,70	73		N	1	Fine.
	3	0	27	45	29,81	69		N	1	Cloudy.
21	8	0	30	42	29,81	74		E	1	Cloudy.
	3	0	32	46	29,81	73		N	1	Snow.
22	8	0	30	42	29,83	74		NE	1	Snow.
	3	0	33	40	29,83	74		N	1	Snow.
23	8	0	30	40	29,88	75		N	1	Cloudy.
	3	0	33	41	29,88	71		W	1	Cloudy.
24	8	0	22	39	29,78	73		W	1	Cloudy.
	3	0	29	43	29,64	73			1	Thick fog.
25	8	0	27	40	29,64	73		E	1	Cloudy.
	3	0	29	44	29,62	73		E	1	Cloudy.
26	8	0	29	40	29,47	74		NE	1	Snow.
	3	0	34	46	29,31	73		NW	1	Cloudy.
27	8	0	28	41	29,00	71		E	2	Snow.
	3	0	32	45	28,95	75		E	1	Cloudy.
28	8	0	35	42	28,94	78		E	1	Fine.
	3	0	38	49	29,01	72		E	1	Cloudy.
29	8	0	45	44	29,11	75		SSE	1	Cloudy.
	3	0	40	44	29,19	74		SE	1	Cloudy.
30	8	0	36	43	29,27	77		E	1	Cloudy.
	3	0	39	47	29,27	72		E	1	Thick and cloudy.
31	8	0	39	49	29,36	78		E	1	Thick and cloudy.
	3	0	43	50	29,38	78		E	1	Cloudy.

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1815	Time. H.	Time. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °		Winds. Points.	Winds. Str.	Weather.
Feb. 1	8	0	44	48	29,38	79	Rain this Month 0,675 Inches.	S	1	Cloudy.
	3	0	45	52	29,40	78		S	1	Cloudy.
2	8	0	40	48	29,52	77		E	1	Cloudy.
	3	0	44	54	29,57	74		E	1	Hazy.
3	8	0	40	51	29,68	78		E	1	Cloudy.
	3	0	44	54	29,65	75		S	1	Cloudy.
4	8	0	40	50	29,57	77			1	Cloudy.
	3	0	49	54	29,51	70		W	1	Cloudy.
5	8	0	44	51	29,90	76		W	1	Fine.
	3	0	47	52	29,97	70		W	1	Fine.
6	8	0	44	52	29,81	75		S	1	Cloudy.
	3	0	45	56	29,69	72		SSE	1	Cloudy.
7	8	0	45	52	29,59	77		S	1	Cloudy.
	3	0	49	56	29,71	69		W	1	Cloudy.
8	8	0	44	53	29,80	77		S	1	Cloudy.
	3	0	47	56	29,69	73		S	1	Cloudy.
9	8	0	40	54	29,71	78		W	1	Cloudy.
	3	0	44	55	29,74	77		SE	1	Cloudy.
10	8	0	39	55	29,69	77		E	1	Cloudy.
	3	0	42	55	29,65	76		W	1	Rain.
11	8	0	45	54	29,51	78		E	1	Cloudy.
	3	0	47	56	29,44	74		S	1	Cloudy.
12	8	0	47	53	29,46	76		SW	1	Cloudy.
	3	0	48	53	29,44	75		W	1	Showery.
13	8	0	47	53	29,46	78		S	1	Cloudy.
	3	0	49	57	29,57	64		W	1	Fine.
14	8	0	40	54	29,56	76		E	1	Hazy.
	3	0	48	57	29,53	71		E	1	Cloudy.
15	8	0	41	51	29,70	76		W	1	Cloudy.
	3	0	49	57	29,73	75		S	1	Cloudy.
16	8	0	48	55	29,64	77		SW	1	Rain.
	3	0	51	57	29,54	73		W	1,2	Cloudy.

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1815	Time.		Therm. without.	Therm. within.	Barom.	Hy-gro-me-ter.		Winds.		Weather.
	H.	M.	o	o	Inches.	o		Points.	Str.	
Feb. 17	8	0	45	56	29,57	73	Rain this Month 0,675 Inches.	W	1	Cloudy.
	3	0	48	58	29,71	71		W	1	Cloudy.
18	8	0	40	53	29,84	73		W	1	Cloudy.
	3	0	45	55	29,81	69		SW	1	Cloudy.
19	8	0	45	53	30,26	75		W	1	Cloudy.
	3	0	48	53	30,20	70		W	1	Cloudy.
20	8	0	47	53	29,77	74		SW	1	Cloudy.
	3	0	48	56	29,70	71		W	1	Fine.
21	8	0	50	55	29,89	76		W	2	Cloudy.
	3	0	54	57	30,04	72		W	1	Cloudy.
22	8	0	48	53	30,20	73		W	1	Cloudy.
	3	0	53	57	30,25	71		W	1	Cloudy.
23	8	0	43	53	30,25	72		W	1	Fine.
	3	0	38	58	30,24	70		SW	1	Fine.
24	8	0	46	56	30,09	71		SW	1	Cloudy.
	3	0	50	58	30,06	68		W	1	Fine.
25	8	0	47	55	30,04	74		W	1	Cloudy.
	3	0	49	58	30,05	71		SW	1	Cloudy.
26	8	0	48	53	29,99	70		NE	1,2	Cloudy.
	3	0	48	55	30,10	72		NW	1	Rain.
27	8	0	38	50	30,43	71		W	1	Hazy.
	3	0	46	57	30,51	63		W	1	Fine.
28	8	0	36	50	30,49	71		W	1	Foggy.
	3	0	43	58	30,41	65		E	1	Fine.

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1815	Time. H.	Time. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °		Winds. Points.	Winds. Str.	Weather.
Mar. 1	8	0	40	52	29,21	75	Rain this Month 1,3250 Inches.	E	1	Hazy.
	3	0	49	59	29,95	71		S	1	Fine.
2	8	0	40	55	30,26	76		SW	1	Cloudy.
	3	0	45	58	30,30	76		N by E	1	Rain.
3	8	0	39	51	30,25	76		W	1	Cloudy.
	3	0	48	58	30,26	64		S	1	Cloudy.
4	8	0	45	53	30,27	75		SW	1	Cloudy.
	3	0	50	57	30,28	68		SW	1	Cloudy.
5	8	0	46	54	30,19	74		SW	1	Cloudy.
	3	0	51	55	30,16	73		SW	1	Cloudy.
6	8	0	42	54	30,23	74		W	1	Cloudy.
	3	0	50	60	30,20	66		S	1	Fine.
7	8	0	46	54	29,96	76		SW	1	Cloudy.
	3	0	49	57	29,85	76		SW	1	Rain.
8	8	0	46	55	29,41	76		W	2	Rain. Gale of wind in the night.
	3	0	48	57	29,40	70		W	1	Rain.
9	8	0	39	48	29,54	72		SW	1	Cloudy.
	3	0	45	56	29,48	65		NW	1	Cloudy.
10	8	0	37	52	29,19	72		W	1	Cloudy
	3	0	42	54	29,29	68		W	1	Cloudy.
11	8	0	34	49	29,38	71		W	1	Cloudy.
	3	0	44	57	29,49	62		WNW	1	Fine.
12	8	0	38	49	29,54	71		S	1	Cloudy.
	3	0	48	55	29,18	76		SSW	1	Rain.
13	8	0	42	50	28,92	69		W	2,3	Cloudy.
	3	0	47	54	28,90	68		W	1,2	Cloudy.
14	8	0	40	49	29,53	70		W	1	Cloudy.
	3	0	45	55	29,75	66		NNE	1	Cloudy.
15	8	0	39	49	30,01	71		W	1	Cloudy.
	3	0	50	55	29,90	77		W	1	Fine.
16	8	0	52	54	29,72	79		W	2	Rain.
	3	0	54	58	29,86	61		N N W	1	Cloudy.

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1815	Time. H.	Time. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hy-gro-me-ter. °		Winds. Points.	Winds. Str.	Weather.
Mar. 17	8	0	46	55	29,97	73	Rain this Month 1,3250 Inches.	W	1	Cloudy.
	3	0	51	58	30,03	59		NNW	1	Fine.
18	8	0	47	53	30,05	70		NW	1	Cloudy.
	3	0	52	57	30,07	66		NW	1	Cloudy.
19	8	0	50	55	29,96	78		W	1	Cloudy.
	3	0	55	55	29,95	66		N	1	Cloudy.
20	8	0	49	55	29,97	74		W by N	1	Cloudy.
	3	0	55	59	29,96	66		SW	1	Cloudy.
21	8	0	49	55	29,82	72		W	1	Cloudy.
	3	0	53	57	29,72	68		S	1	Cloudy.
22	8	0	49	55	29,65	73		SW	1,2	Cloudy.
	3	0	54	59	29.57	70		SW	1	Rain.
23	8	0	52	56	29,37	73		W	1	Fine.
	3	0	52	59	29,24	60		W by N	2	Showery.
24	8	0	46	53	29,49	68		W	1	Cloudy.
	3	0	51	57	29,48	71		W	1,2,3	Rain. Squally.
25	8	0	44	54	29,31	71		S	1	Fine.
	3	0	48	56	29,46	65		W	1	Fine.
26	8	0	42	49	29,77	68		SW	1	Cloudy.
	3	0	47	55	29,71	70		SSW	2	Cloudy and squally.
27	8	0	48	52	29,51	75		SW	2,3	Cloudy and squally.
	3	0	53	56	29,49	72		SW	1	Rain.
28	8	0	52	54	29,56	73		W	2	Cloudy.
	3	0	55	58	29,83	58		NW	1	Fine.
29	8	0	49	54	30,03	70		S by E	1	Cloudy.
	3	0	58	62	29,93	63		S	1,2	Fine.
30	8	0	48	56	30,01	74		W	1	Cloudy.
	3	0	55	59	29,97	70		WSW	1	Cloudy.
31	8	0	47	55	29,95	74		E	1	Hazy.
	3	0	63	63	29,89	58		E	1	Fair.

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1815	Time. H.	M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °		Winds. Points.	Str.	Weather.
April 1	7	30	55	59	29,72	70	Rain this Month 1,660 Inches.	ENE	1	Fair.
	3	0	63	67	29,67	56		S	2	Fair.
2	7	30	54	60	29,81	70		W	1	Cloudy.
	3	0	58	64	29,90	60		W	1	Fine.
3	7	30	49	57	29,86	70		W	1	Hazy.
	3	0	54	62	29,88	60		NW	1	Fine.
4	7	30	43	57	29,98	69		SW	1	Fine.
	3	0	57	60	29,96	58		W	1	Cloudy.
5	7	30	42	55	30,24	66		NW	1	Cloudy.
	3	0	54	58	30,27	60		NW	1	Fine.
6	7	30	48	53	30,22	70		NW	1	Fine.
	3	0	59	62	30,19	65		W	1	Cloudy.
7	7	30	55	57	30,15	73		S	1	Fine.
	3	0	60	65	30,07	61		E	1	Fair.
8	7	30	51	57	30,01	72		E	1	Cloudy.
	3	0	56	62	29,94	64		E	1	Fine.
9	7	30	52	56	29,88	69		E	1	Cloudy.
	3	0	56	59	29,88	68		E	1	Cloudy.
10	7	30	46	56	29,91	76		NE	1	Rain.
	3	0	53	57	29,92	79		ESE	1	Cloudy.
11	7	0	50	57	30,00	77		SW	1	Cloudy.
	3	0	59	61	30,00	65		S	1	Fine.
12	7	0	53	57	29,97	73		NE	1	Hazy.
	3	0	60	60	29,91	64		E	1	Cloudy.
13	7	0	48	55	29,86	72		W	1	Hazy.
	3	0	56	59	29,69	68		W	1	Rain and thunder.
14	7	0	42	51	29,67	68		N	1	Cloudy.
	3	0	47	52	29,87	65		NNE	1	Cloudy.
15	7	0	41	52	30,07	65		NE	1	Cloudy.
	3	0	45	54	30,11	60		N	1	Cloudy.
16	7	0	40	51	30,16	64		N	1	Cloudy.
	3	0	49	53	30,15	56		N	1	Cloudy.

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1815	Time. H.	Time. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °		Winds. Points.	Winds. Str.	Weather.
Apr. 17	7	0	43	50	30,17	67	Rain this Month 1,864 Inches.	N	1	Fine.
	3	0	48	53	30,19	60		NE	1	Cloudy.
18	7	0	40	50	30,32	70		N	1	Fine, rather hazy.
	3	0	50	54	30,31	58		NNE	1	Cloudy.
19	7	0	42	50	30,26	66		NNE	1	Fine.
	3	0	50	53	30,13	60		N	1	Cloudy.
20	7	0	45	51	29,92	69		N	1	Cloudy.
	3	0	50	55	29,81	63		N	1	Cloudy.
21	7	0	43	52	29,35	69		S	1	Cloudy.
	3	0	45	55	28,92	71		E	2	Rain.
22	7	0	43	51	28,78	77		E	2	Cloudy.
	3	0	45	54	28,92	71		N	1	Cloudy.
23	7	0	45	51	29,11	72		Variable.	1	Cloudy.
	3	0	45	52	29,22	73		N	1	Rain.
24	7	0	42	50	29,30	72		NE	1	Cloudy.
	3	0	47	53	29,50	73		N	1	Cloudy.
25	7	0	40	50	29,68	70		WNW	1	Fine.
	3	0	48	54	29,71	61		NE	1	Cloudy.
26	7	0	46	52	29,94	70		E	1	Cloudy.
	3	0	50	58	30,09	60		E	1	Fine.
27	7	0	45	52	30,13	74		NE	1	Cloudy.
	3	0	54	56	30,07	68		E	1	Cloudy.
28	7	0	49	55	30,60	67		N	1	Cloudy.
	3	0	58	60	29,91	55		NE	1	Cloudy.
29	7	0	49	55	29,73	68		Variable.	1	Cloudy.
	3	0	50	57	29,70	67		E	1	Cloudy.
30	7	0	46	54	29,59	65		E	1	Cloudy.
	3	0	50	54	29,58	71		E	1	Rain.

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1815	Time.		Therm. without.	Therm. within.	Barom.	Hy-gro-me-ter.		Winds.		Weather.
	H.	M.	o	o	Inches.	o		Points.	Str.	
May 1	7	0	51	55	29,71	75	Rain this Month 0,667 Inches.	E	1	Fine.
	3	0	60	60	29,72	65		E	1	Cloudy.
2	7	0	55	56	29,74	70		E	1	Fine.
	3	0	60	59	29,75	67		E	1	Cloudy, much thunder at a distance.
3	7	0	51	56	29,78	75		NE	1	Cloudy.
	3	0	64	60	29,73	61		E	1	Fine.
4	7	0	50	57	29,77	74		NE	1	Cloudy.
	3	0	60	61	29,74	68		NE	1	Rain, and thunder at a distance.
5	7	0	53	59	29,75	73		W	1	Fine.
	3	0	60	61	29,75	64		W	1	Fine.
6	7	0	50	58	29,78	75		W	1	Thick and cloudy.
	3	0	61	61	29,78	63		SW	1	Fine.
7	7	0	50	59	29,76	66		S	1	Cloudy.
	3	0	62	60	29,75	65		SW	1	Cloudy.
8	7	0	54	59	29,75	72		S	1	Cloudy.
	3	0	62	64	29,74	62		SW	1	Cloudy.
9	7	0	53	59	29,88	68		W	1	Cloudy.
	3	0	63	62	29,96	58		NE	1	Cloudy.
10	7	0	53	59	30,00	73		SW	1	Rain.
	3	0	62	61	29,98	67		SW	1	Cloudy.
11	7	0	57	59	29,91	69		SW	1	Fine.
	3	0	66	65	29,82	60		S	1	Fine.
12	7	0	56	60	29,63	66		W	2	Cloudy.
	3	0	62	62	29,68	61		SW	2	Cloudy.
13	7	0	55	60	29,75	69		S	1,2	Cloudy.
	3	0	62	65	29,73	61		SW	1	Fine.
14	7	0	51	59	29,83	68		W	1	Fine.
	3	0	60	62	29,88	60		W	1,2	Fine.
15	7	0	53	58	29,87	67		S	1	Cloudy.
	3	0	61	60	29,83	61		S	1,2	Cloudy.
16	7	0	50	58	30,01	70		SW	1	Fine.
	3	0	61	62	30,11	57		W	1	Cloudy.

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1815	Time.		Therm. without.	Therm. within.	Barom.	Hygrometer.		Winds.		Weather.
	H.	M.	°	°	Inches.			Points.	Str.	
May 17	7	0	53	58	30,32	66	Rain this Month 0,667 Inches.	W	1	Cloudy.
	3	0	64	62	30,33	56		W	1	Cloudy.
18	7	0	59	60	30,30	72		W	1	Cloudy and hazy.
	3	0	65	62	30,29	64		NW	1	Cloudy.
19	7	0	60	61	30,23	68		N	1	Cloudy.
	3	0	65	62	30,17	62		N	1	Cloudy.
20	7	0	58	60	29,96	68		W	1	Fine.
	3	0	61	64	29,74	65		W	1	Cloudy.
21	7	0	54	60	29,62	62		W	1	Cloudy.
	3	0	63	63	29,62	58		N	1	Fine.
22	7	0	52	56	29,85	64		N	1	Cloudy.
	3	0	59	59	29,92	58		N	1	Cloudy
23	7	0	50	57	29,97	64		SW	1	Cloudy.
	3	0	59	58	29,84	61		W	1	Rain.
24	7	0	51	56	29,88	74		W	1	Rain.
	3	0	61	60	29,90	64		WNW	1	Cloudy.
25	7	0	56	58	30,07	71		W	1	Cloudy.
	3	0	66	64	30,10	59		W	1	Fine.
26	7	0	58	59	30,22	71		W	1	Cloudy.
	3	0	68	64	30,22	57		N	1	Fine.
27	7	0	57	60	30,22	69		E	1	Fine.
	3	0	62	67	30,15	58		E	1	Fine.
28	7	0	60	61	29,99	65		E	1	Fair.
	3	0	66	69	29,93	61		E	1	Fine.
29	7	0	61	63	29,85	69		E	1	Cloudy and hazy.
	3	0	59	63	29,84	66		SW	1	Cloudy.
30	7	0	54	60	29,88	66		S	1	Cloudy.
	3	0	63	63	29,89	60		N	1	Cloudy.
31	7	0	53	60	29.92	68		W	1	Cloudy.
	3	0	63	63	29,87	58		SW	1	Fine.

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1815	Time. H.	Time. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °		Winds. Points.	Winds. Str.	Weather.
June 1	7	0	55	60	29,79	72	Rain this Month 1,752 Inches.	E	1	Cloudy.
	3	0	62	63	29,89	70		E	1	Cloudy.
2	7	0	55	57	30,11	66		W	1	Fine.
	3	0	67	64	30,08	57		W	1	Fine.
3	7	0	57	59	30,02	70		W	1	Rain.
	3	0	65	63	29,97	65		W	1	Cloudy.
4	7	0	61	62	29,93	68		W	1	Fine.
	3	0	68	67	29,82	61		W	1	Fine.
5	7	0	57	62	29,67	68		W	1	Cloudy.
	3	0	63	64	29,59	64		NW	1	Cloudy.
6	7	0	57	59	29,53	64		S	1	Cloudy.
	3	0	60	62	29,45	68		N	1	Rain.
7	7	0	53	58	29,53	68		NW	1	Cloudy.
	3	0	62	62	29,66	59		S	1	Fine.
8	7	0	57	60	29,70	64		N	1	Cloudy.
	3	0	63	62	29,74	59		E	1	Cloudy.
9	7	0	59	60	29,85	60		E	1	Fine.
	3	0	65	64	29,85	57		E	1	Fine.
10	7	0	58	59	29,87	63		NW	1	Fine.
	3	0	66	65	29,89	56		N N W	1	Fine.
11	7	0	58	62	29,87	66		SW	1	Fine.
	3	0	67	66	29,81	56		SW	1	Cloudy.
12	7	0	55	61	29,84	67		E	1	Cloudy.
	3	0	62	64	29,77	59		SE	1	Fine.
13	7	0	52	61	29,59	59		E	1	Rain.
	3	0	61	64	29,53	64		W	1	Cloudy.
14	7	0	55	60	29,30	74		S	1	Cloudy.
	3	0	62	64	29,32	62		S	1	Fine.
15	7	0	57	61	29,49	71		W	1	Cloudy.
	3	0	66	66	29,67	57		S	1	Cloudy.
16	7	0	59	60	29,88	63		SSE	1	Fine.
	3	0	68	66	29,82	57		SW	1	Fine.

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1815	Time.		Therm. without.	Therm. within.	Barom.	Hy-gro-me-ter.		Winds.		Weather.
	H.	M.	°	°	Inches.	°		Points.	Str.	
June 17	7	0	61	63	29,61	82	Rain this Month 0,752 Inches.	E	1	Cloudy, much rain in the [night.
	3	0	67	65	29,59	58		W	1	Cloudy.
18	7	0	58	63	29,61	62		W	1	Cloudy.
	3	0	67	65	29,62	53		W	1	Cloudy.
19	7	0	62	61	29,72	58		W	1	Fine.
	3	0	67	64	29,71	53		W	1	Cloudy.
20	7	0	62	63	29,71	60		E	1	Cloudy.
	3	0	67	67	29,70	54		S	1,2	Cloudy.
21	7	0	60	63	29,77	62		N	1,2	Fine.
	3	0	64	65	29,84	69		E	1	Cloudy.
22	7	0	58	60	29,92	66		W	1	Cloudy and hazy.
	3	0	66	66	29,89	62		N	1	Cloudy.
23	7	0	58	61	29,92	62		W	1	Cloudy.
	3	0	65	65	30,00	65		W	1	Fine.
24	7	0	59	62	30,07	60		NW	1	Cloudy.
	3	0	67	67	30,06	53		W	1	Fine.
25	7	0	59	63	29,98	64		N	1	Cloudy.
	3	0	61	64	30,05	56		N	1	Cloudy.
26	7	0	58	60	30,11	62		N	1	Cloudy.
	3	0	66	65	30,10	52		N	1	Cloudy.
27	7	0	56	62	30,11	58		N	1	Fine, rather hazy.
	3	0	68	68	30,11	54		E	1	Fine.
28	7	0	62	63	30,20	60		E	1	Cloudy.
	3	0	68	70	30,24	54		E	1	Fair.
29	7	0	60	65	30,29	59		E	1	Cloudy.
	3	0	69	70	30,27	50		E	1	Fair.
30	7	0	60	64	30,24	60		E	1	Cloudy.
	3	0	70	70	30,20	57		E	1	Cloudy.

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1815	Time.		Therm. without.	Therm. within.	Barom.	Hygrometer.		Winds.		Weather.
	H.	M.	°	°	Inches.	°		Points.	Str.	
July 1	7	0	60	64	30,14	66	Rain this Month 1,581 Inches.	N	1	Cloudy and hazy.
	3	0	71	72	30,11	56		N	1	Fine.
2	7	0	58	65	30,15	62		N	1	Cloudy.
	3	0	67	68	30,09	53		N	1	Fine.
3	7	0	58	62	30,06	57		NE	1	Cloudy.
	3	0	65	66	30,01	52		N	1	Cloudy.
4	7	0	60	62	30,01	62		E	1	Cloudy.
	3	0	64	65	30,01	57		E	1	Cloudy.
5	7	0	59	62	30,08	57		NNW	1	Cloudy.
	3	0	66	67	30,09	52		N	1	Fine.
6	7	0	59	63	30,08	59		W	1	Cloudy.
	3	0	66	67	29,99	51		N	1	Fair.
7	7	0	55	62	30,08	64		N	1	Cloudy.
	3	0	60	65	30,03	52		N	1	Cloudy.
8	7	0	56	61	30,12	59		NW	1	Fine.
	3	0	63	64	30,12	50		NW	1	Cloudy.
9	7	0	62	62	30,06	60		N	1	Cloudy.
	3	0	63	64	30,11	56		NE	1	Cloudy.
10	7	0	60	62	30,19	59		NW	1	Fine.
	3	0	62	67	30,17	58		W	1	Fine.
11	7	0	60	63	30,19	59		N	1	Fine.
	3	0	68	70	30,16	53		W	1	Fair.
12	7	0	62	64	30,12	60		S	1	Fair.
	3	0	70	69	30,07	53		S	1	Fine.
13	7	0	62	64	30,06	60		S	1	Fair.
	3	0	70	69	30,02	51		W	1	Fine.
14	7	0	63	65	30,09	61		W	1	Cloudy.
	3	0	72	70	30,08	55		SW	1	Cloudy.
15	7	0	67	67	30,06	61		SW	1	Fine.
	3	0	71	70	30,00	57		SW	2	Cloudy.
16	7	0	64	66	29,98	62		W	1	Cloudy.
	3	0	71	69	30,05	55		NW	1	Cloudy.

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1815	Time.		Therm. without.	Therm. within.	Barom.	Hy-gro-me-ter.		Winds.		Weather.
	H.	M.	°	°	Inches.	°		Points.	Str.	
July 17	7	0	63	67	30,00	62	Rain this Month 0,581 Inches.	W	1	Cloudy.
	3	0	71	69	29,86	57		W	2	Cloudy.
18	7	0	58	66	29,82	61		W	2	Cloudy.
	3	0	70	70	29,83	48		W	1	Fine.
19	7	0	56	65	29,76	62		SW	1	Rain.
	3	0	62	66	29,62	67		W	1	Rain.
20	7	0	57	64	29,73	63		N	1	Cloudy.
	3	0	62	65	29,82	56		N	1	Cloudy.
21	7	0	56	63	29,94	62		N	1	Fine.
	3	0	64	68	29,94	52		NW	1	Fine.
22	7	0	58	63	29,95	58		NNE	1	Cloudy.
	3	0	65	65	29,92	52		N	1	Cloudy.
23	7	0	57	62	29,93	59		N	1	Cloudy.
	3	0	65	65	29,92	54		N	1	Cloudy
24	7	0	59	63	29,99	60		N	1	Fine.
	3	0	66	65	30,01	52		NE	1	Cloudy.
25	7	0	62	64	30,00	59		E	1	Fine.
	3	0	67	66	30,11	56		NW	1	Cloudy.
26	7	0	62	64	30,09	65		NE	1	Rain.
	3	0	65	65	30,15	63		SW	1	Cloudy.
27	7	0	56	62	30,19	61		NE	1	Fair.
	3	0	61	63	30,21	58		NEb.E	1	Cloudy.
28	7	0	57	61	30,18	59		NNE	1	Fair.
	3	0	65	67	30,17	54		SE	1	Fair.
29	7	0	61	63	30,16	60		W	1	Fine.
	3	0	68	65	30,09	53		SW	1	Cloudy.
30	7	0	68	64	30,02	58		S	1	Fair.
	3	0	70	66	30,05	64		NE	1	Fair.
31	7	0	58	54	30,12	63		SSW	1	Cloudy.
	3	0	61	64	30,15	57		SE	1	Cloudy.

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1815	Time.		Therm. without.	Therm. within.	Barom.	Hygrometer.		Winds.		Weather.
	H.	M.	°	°	Inches.	°		Points.	Str.	
Aug. 1	7	0	57	62	30,23	59	Rain this Month 0,222 Inches.	NW	1	Cloudy.
	4	0	64	51	30,27	57		NE	1	Cloudy.
2	7	0	62	61	30,22	65		SE	1	Cloudy.
	4	0	67	66	30,18	54		NNE	1	Fair.
3	7	0	64	65	30,21	69		SW	1	Fair.
	4	0	61	63	30,17	70		SSE	1	Cloudy.
4	7	0	51	64	30,08	69		NE	1	Fair.
	4	0	63	69	30,04	65		SW	1	Cloudy.
5	7	0	61	65	29,79	70		SSW	1	Fine.
	4	0	60	61	29,77	69		SW	1	Fair.
6	7	0	55	63	29,71	63		W	1	Fair.
	4	0	67	66	29,68	69		W	1	Fine.
7	7	0	60	62	29,69	66		W	1	Fair.
	4	0	69	63	29,81	63		S	1	Fair.
8	7	0	59	57	29,86	65		SW	1	Cloudy.
	4	0	62	61	29,81	67		W	1	Cloudy.
9	7	0	65	65	29,86	69		S	1	Fair.
	4	0	68	69	29,81	66		SSE	1	Fair.
10	7	0	67	63	29,89	65		SSW	1	Fine.
	4	0	65	63	29,89	66		SW	1	Cloudy.
11	7	0	66	64	29,56	69		SE	1	Fine.
	4	0	62	69	29,56	68		SW	1	Fair.
12	7	0	68	66	29,57	65		SSE	1	Fine.
	4	0	69	66	29,57	68		SW	1	Cloudy.
13	7	0	66	62	29,78	66		SSW	1	Fair.
	4	0	68	67	29,78	68		SW	1	Fine.
14	7	0	65	66	29,87	59		S	1	Cloudy.
	4	0	69	68	29,89	63		W	1	Cloudy.
15	7	0	64	62	30,07	65		SW	1	Cloudy.
	4	0	62	51	30,08	62		E	1	Fair.
16	7	0	68	61	30,11	65		SE	1	Cloudy.
	4	0	69	66	30,08	67		W	1	Cloudy.

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1815	Time. H.	Time. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °		Winds. Points.	Winds. Str.	Weather.
Aug. 17	7	0	65	63	30,07	66	Rain this Month 0,222 Inches.	SE	1	Fair.
	4	0	68	67	30,01	68		SE	1	Cloudy.
18	7	0	62	67	29,97	66		SE	1	Fair.
	4	0	62	65	29,91	69		W	1	Cloudy.
19	7	0	68	66	29,97	66		W	1	Fair.
	4	0	66	54	29,98	65		SW	1	Cloudy.
20	7	0	60	66	30,01	64		SW	1	Cloudy.
	4	0	63	65	30,08	62		W	1	Cloudy.
21	7	0	54	54	29,97	59		SW	1	Fine.
	4	0	64	56	29,91	57		SSE	1	Fine.
22	7	0	52	56	29,88	58		W	1	Fair.
	4	0	69	68	29,81	68		S	1	Cloudy.
23	7	0	66	58	29,76	62		S	1	Cloudy.
	4	0	55	63	29,78	67		SW	1	Cloudy. Rain in the night.
24	7	0	63	52	29,98	66		SE	1	Fair.
	4	0	52	62	30,01	61		W	1	Cloudy.
25	7	0	65	64	30,11	54		SSW	1	Cloudy.
	4	0	68	66	30,17	52		SW	1	Fair.
26	7	0	63	65	30,01	68		SE	1	Fine.
	4	0	65	63	30,01	65		SW	1	Fair.
27	7	0	62	67	29,99	64		E	1	Cloudy.
	4	0	63	65	29,92	68		W	1	Cloudy.
28	7	0	64	59	29,99	67		S	1	Fair.
	4	0	65	67	29,92	62		SSW	1	Fine.
29	7	0	68	56	29,99	61		SW	1	Cloudy.
	4	0	69	69	29,98	66		S	1	Cloudy.
30	7	0	67	58	30,05	59		SE	1	Hazy.
	4	0	62	61	30,18	54		SW	1	Cloudy.
31	7	0	66	61	30,17	52		S	1	Fine.
	4	0	61	64	30,11	51		W	1	Fine.

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1815	Time. H.	Time. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer.		Winds. Points.	Winds. Str.	Weather.
Sep. 1	7	0	68	62	30,19	62°	Rain this Month 0,776 Inches.	E	1	Cloudy.
	3	0	61	69	30,19	66		NE	1	Fair.
2	7	0	65	59	29,99	65		W	1	Fine.
	3	0	67	67	30,01	62		SSW	1	Fine.
3	7	0	66	63	30,03	64		W	1	Fine.
	3	0	69	65	30,03	68		NW	1	Fair,
4	7	0	68	67	30,08	63		E	1	Cloudy.
	3	0	65	64	30,01	58		W	1	Cloudy.
5	7	0	67	62	29,99	59		NNW	1	Cloudy.
	3	0	60	63	29,98	64		N	1	Cloudy.
6	7	0	62	61	30,19	62		NE	1	Cloudy.
	3	0	69	62	30,19	65		W	1	Fine.
7	7	0	65	65	30,18	69		N	1	Fair.
	3	0	63	69	30,17	68		NNW	1	Fair.
8	7	0	61	64	30,17	63		S	1	Fine.
	3	0	68	68	30,11	67		SSE	1	Cloudy.
9	7	0	67	63	30,19	69		W	1, 2	Fine.
	3	0	64	61	30,19	67		NWbW	1	Fine.
10	7	0	66	68	30,19	68		E	1	Fine.
	3	0	65	58	30,20	69		N	1	Cloudy.
11	7	0	69	54	30,20	62		W	1	Fine.
	3	0	68	69	30,16	69		E	1	Cloudy.
12	7	0	54	59	30,17	68		N	1	Cloudy.
	3	0	61	59	30,19	62		W	1	Cloudy.
13	7	0	69	68	29,99	67		E	1	Fine.
	3	0	65	63	29,99	61		N	1	Fine.
14	7	0	68	64	29,88	65		NW	1	Fine.
	3	0	63	66	29,89	69		E	1	Cloudy.
15	7	0	62	61	29,89	59		N	1	Cloudy.
	3	0	69	69	29,81	58		NE	1	Cloudy.
16	7	0	69	59	29,83	56		E	1	Cloudy.
	3	0	67	62	29,81	62		NE	1	Fair.

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1815	Time. H.	Time. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °		Winds. Points.	Winds. Str.	Weather.
Sept. 17	7	0	67	56	29,90	65	Rain this Month 0,776 Inches.	W	1	Cloudy.
	3	0	63	69	29,99	69		N	1	Cloudy.
18	7	0	66	54	30,06	66		NNE	1	Cloudy.
	3	0	65	68	30,19	63		W	1	Fair.
19	7	0	70	57	30,11	59		S	1	Fine.
	3	0	62	69	30,13	57		SW	1	Fine.
20	7	0	68	59	29,98	60		SSE	1	Fine.
	3	0	68	62	29,99	61		N	1	Fine.
21	7	0	66	58	29,98	63		NE	1	Cloudy.
	3	0	67	66	29,97	62		E	1	Cloudy.
22	7	0	61	61	29,79	68		W	1	Cloudy.
	3	0	63	62	29,77	69		N	1	Cloudy.
23	7	0	59	67	29,71	67		NW	1	Fair.
	3	0	61	61	29,78	63		NW	1	Fair.
24	7	0	66	68	29,79	62		NW	1	Cloudy.
	3	0	59	66	29,79	61		W	1	Cloudy.
25	7	0	68	64	29,81	68		S	1	Fine.
	3	0	60	68	29,86	67		SW	1	Fine.
26	7	0	59	63	29,78	65		NW	1	Fine.
	3	0	58	64	29,77	64		NW	1	Fine.
27	7	0	60	69	29,91	66		NNE	1	Fine.
	3	0	63	62	29,98	63		E	1	Fair.
28	7	0	67	70	29,97	61*		N	1	Fair.
	3	0	64	67	29,99			S	1	Cloudy.
29	7	0	66	66	29,49			NW	1	Cloudy.
	3	0	62	63	29,49			W	1	Fine.
30	7	0	61	62	29,57			WNW	1	Cloudy.
	3	0	63	65	29,59			E	1	Fine.

* The Hygrometer was broken by the workmen employed in repairing the building.

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1815	Time.		Therm. without.	Therm. within.	Barom.	Hygrometer.		Winds.		Weather.
	H.	M.	°	°	Inches.	°		Points.	Str.	
Oct. 1	7	0	59	62	29,77		Rain this Month 1,889 Inches.	W	1	Fair.
	3	0	63	59	29,86			E	1	Fair.
2	7	0	53	49	30,16			S	1	Fine, rather foggy.
	3	0	55	52	30,19			N	1	Cloudy.
3	7	0	57	54	30,02			SE	1	Cloudy.
	3	0	57	57	30,03			S	1	Cloudy.
4	7	0	56	53	30,03			S	1	Fine.
	3	0	59	56	30,04			SE	1	Fine.
5	7	0	57	51	30,04			E	1	Hazy.
	3	0	60	60	29,97			S	1	Cloudy.
6	7	0	58	60	29,89			SE	1	Rain.
	3	0	59	60	30,05			E	1	Cloudy.
7	7	0	49	57	30,16			N	1	Fine.
	3	0	57	62	30,18			NNE	1	Fine.
8	7	0	51	57	30,28			N	1	Hazy.
	3	0	54	60	30,29			E	1	Fine.
9	7	0	48	56	30,28			E	1	Fine.
	3	0	52	59	30,21			E	1	Fine.
10	7	0	48	56	30,11			E	1	Fine.
	3	0	53	61	30,00			E	1	Fine.
11	7	0	49	56	29,82			E	1	Cloudy.
	3	0	52	57	29,75			E	1	Rain.
12	7	0	48	55	29,70			NE	1	Cloudy.
	3	0	52	57	29,71			NNW	1	Cloudy.
13	7	0	47	54	29,78			E	1	Cloudy.
	3	0	56	59	29,80			SE	1	Rain.
14	7	0	55	56	29,72			W	1	Fine.
	3	0	55	58	29,76			W	1	Fine.
15	7	0	53	57	29,80			W	1	Fine.
	3	0	57	58	29,92			W	1	Cloudy.
16	7	0	56	58	29,86			WSW	1	Fine.
	3	0	59	63	29,87			W	1	Fine.

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1815	Time.		Therm. without.	Therm. within.	Barom.	Hygrometer.		Winds.		Weather.
	H.	M.	°	°	Inches.			Points.	Str.	
Oct. 17	7	0	52	59	29,82		Rain this Month 1,889 Inches.	W	1	Fair.
	3	0	54	62	29,87			W	1	Fair.
18	7	0	44	57	29,94			S	1	Rain.
	3	0	54	60	29,85			W	1	Rain.
19	7	0	53	58	29,59			S	1	Rain.
	3	0	61	63	29,44			WSW	2,3	Cloudy.
20	7	0	59	61	29,33			S	2	Fine.
	3	0	59	63	29,42			SW	1	Fine.
21	7	0	53	60	29,54			WNW	1	Fine.
	3	0	56	63	29,65			NW	1	Fine.
22	7	0	46	57	29,88			W	1	Cloudy.
	3	0	55	60	29,93			S	1	Fine.
23	7	0	56	59	29,71			S	1	Rain.
	3	0	55	60	29,63			S	1	Rain.
24	7	0	53	58	29,57			W	2	Fine.
	3	0	56	62	29,55			SW	1	Fine.
25	7	0	49	58	29,53			S	1	Fine.
	3	0	54	62	29,45			SW	1	Fine.
26	7	0	44	57	29,47			S	1	Fine.
	3	0	52	61	29,50			W	1	Hazy.
27	7	0	47	57	29,53			N	1	Rain.
	3	0	49	58	29,52			E	1	Hazy and cloudy.
28	7	0	48	56	29,74			N	1	Rain.
	3	0	50	60	29,92			E	1	Fine.
29	7	0	48	56	30,05			N	1	Fair.
	3	0	53	58	29,99			N	1	Cloudy.
30	7	0	49	55	29,91			NW	1	Fine.
	3	0	50	57	29,92			N	1	Cloudy.
31	7	0	46	55	29,99			N	1	Fine.
	3	0	48	58	30,03			E	1	Fine.

METEOROLOGICAL JOURNAL

for November, 1815.

1815	Time. H.	Time. M.	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °		Winds. Points.	Winds. Str.	Weather.
Nov. 1	8	0	43	54	30,03		Rain this Month 1,336 Inches.	W	1	Cloudy and hazy.
	3	0	48	57	30,04			NE	1	Cloudy.
2	8	0	45	53	30,03			W	1	Hazy.
	3	0	46	55	30,20			NW	1	Hazy.
3	8	0	39	51	30,28			N	1	Fine.
	3	0	45	48	30,34			NW	1	Cloudy.
4	8	0	36	48	30,49			W	1	Foggy.
	3	0	43	50	30,37			W	1	Fine.
5	8	0	43	49	30,32			W	1	Cloudy.
	3	0	48	51	30,28			W	1	Cloudy.
6	8	0	44	49	30,24			W	1	Cloudy.
	3	0	50	50	30,19			W	1	Cloudy.
7	8	0	49	51	30,13			W	1	Rain.
	3	0	48	52	30,12			NW	1	Fine.
8	8	0	42	50	30,10			W	1	Rain.
	3	0	50	55	30,03			W	1	Cloudy.
9	8	0	52	54	29,83			W	1	Rain.
	3	0	54	58	29,93			W	1	Cloudy.
10	8	0	49	56	30,11			W	1	Cloudy.
	3	0	54	58	30,15			W	1	Cloudy.
11	8	0	50	56	30,17			W	1	Cloudy.
	3	0	53	58	30,11			W	1	Cloudy.
12	8	0	51	56	29,96			W	1	Cloudy.
	3	0	54	57	29,84			SW	1	Cloudy.
13	8	0	48	56	29,06			SW	2	Rain.
	3	0	45	58	29,05			W	2	Fine.
14	8	0	39	52	29,14			W	1	Fine.
	3	0	45	57	29,15			W	1	Fine.
15	8	0	39	52	29 06			N	1	Cloudy.
	3	0	42	55	29,17			N	1	Fine.
16	8	0	36	51	29,33			NW	1	Cloudy.
	3	0	38	53	29,48			W	1	Fair.

METEOROLOGICAL JOURNAL

for November, 1815.

1815	Time.		Therm. without.	Therm. within.	Barom.	Hygrometer.	Rain.	Winds.		Weather.
	H.	M.	°	°	Inches.	°		Points.	Str.	
Nov. 17	8	0	31	49	29,55		Rain this Month 1,336 Inches.	W	1	Foggy.
	3	0	38	53	29,70			NW	1	Fine.
18	8	0	30	48	29,94			W	1	Cloudy.
	3	0	38	52	30,02			W	1	Fine.
19	8	0	29	46	30,09			W	1	Fine.
	3	0	35	45	30,09			N	1	Cloudy.
20	8	0	34	44	29,77			N	1	Fine.
	3	0	40	48	29,68			NE	1	Fine.
21	8	0	35	46	29,67			N	1	Fine.
	3	0	42	48	29,69			N	1	Cloudy.
22	8	0	32	46	29,83			N	1	Cloudy.
	3	0	36	50	29,89			N	1	Cloudy.
23	8	0	32	45	30,14			N	1	Fine.
	3	0	40	50	30,18			N	1	Fair.
24	8	0	34	47	30,21			N	1	Hazy.
	3	0	41	49	30,38			NE	1	Rain.
25	8	0	37	47	30,49			N	1	Foggy.
	3	0	43	47	30,61			NE	1	Hazy.
26	8	0	37	46	30,61			N	1	Foggy.
	3	0	42	45	30,50			NE	1	Thick and hazy.
27	8	0	36	46	30,36			NE	1	Thick and hazy.
	3	0	37	48	30,27			E	1	Fine.
28	8	0	33	46	30,00			N	1	Thick and hazy.
	3	0	35	47	30,07			NW	1	Cloudy.
29	8	0	29	45	30,05			W	1	Hazy.
	3	0	32	45	30,00			S	1	Cloudy.
30	8	0	37	45	29,93			ESE	1	Rain.
	3	0	40	48	29,90			S	1	Rain.

METEOROLOGICAL JOURNAL

for December, 1815.

1815	Time.		Therm. without.	Therm. within.	Barom.	Hygrometer.		Winds.		Weather.
	H.	M.	°	°	Inches.	°		Points.	Str.	
De. 1	8	0	48	49	29,81		Rain this Month 0,650 Inches.	S	1	Rain.
	3	0	48	53	29,91			W	2	Cloudy.
2	8	0	46	52	30,11			W	1	Fine.
	3	0	48	54	30,13			W	1	Cloudy.
3	8	0	47	51	30,03			S	1	Cloudy.
	3	0	48	51	30.00			W	1	Fine.
4	8	0	44	50	29,79			S	1	Rain.
	3	0	43	52	29,78			W	1	Fine.
5	8	0	39	50	29.96			W	1	Cloudy and hazy.
	3	0	42	52	29,72			W	1	Rain.
6	8	0	39	50	29,39			W	1	Fine.
	3	0	43	53	29,38			NW	1	Cloudy.
7	8	0	37	49	29,78			E	1	Cloudy.
	3	0	35	50	29.94			N	2	Cloudy
8	8	0	27	46	30,00			N	1	Fine.
	3	0	31	50	30,00			N	1	Fair.
9	8	0	26	44	30,07			N	1	Fine.
	3	0	32	48	30,14			N	1	Fine
10	8	0	32	43	30,37			NNE	1	Cloudy.
	3	0	36	44	30,38			N	1	Cloudy.
11	8	0	34	42	30,38			N	1	Cloudy and hazy.
	3	0	37	45	30,40			W	1	Cloudy.
12	8	0	36	44	30,27			N	1	Fine, rather hazy.
	3	0	38	48	30 40			W	1	Cloudy.
13	8	0	37	44	30,29			W	1	Cloudy.
	3	0	43	49	30,22			W	1	Rain.
14	8	0	32	49	30,32			W	1	Fine.
	3	0	38	49	30,29			W	1	Fine.
15	8	0	45	48	29,83			W	2	Cloudy.
	3	0	46	52	29,74			W	1	Cloudy.
16	8	0	44	50	29,04			W	1	Cloudy.
	3	0	42	52	29.05			W	1	Fine.

METEOROLOGICAL JOURNAL
for December, 1815.

1815	Time. H.	M	Therm. without. °	Therm. within. °	Barom. Inches.	Hygrometer. °		Points.	Str.	Weather.
Dec. 17	8	0	34	45	28,95		Rain this Month 0,861 Inches.	W	1	Cloudy.
	3	0	36	47	29,08			WNW	2	Cloudy.
18	8	0	29	43	29,39			W	1	Hazy.
	3	0	37	49	29,39			W	1	Fair.
19	8	0	28	44	29,59			W	1	Fine.
	3	0	33	47	29,58			SSE	1	Snow.
20	8	0	44	46	29,19			S	2	Rain.
	3	0	43	49	28,98			W	2,3	Rain.
21	8	0	37	46	29,32			W	1	Cloudy.
	3	0	38	49	29,45			N	1	Rain.
22	8	0	37	41	29,60			W	1	Fine, rather hazy.
	3	0	37	50	29,67			W	1	Cloudy.
23	8	0	32	46	29,76			W	1	Cloudy.
	3	0	40	49	29,53			W	1	Cloudy.
24	8	0	41	47	29,35			W	1	Fine.
	3	0	42	48	29,33			W	1	Fine.
25	8	0	33	44	29,53			W	1	Hazy.
	3	0	34	49	29,76			SE	1	Cloudy.
26	8	0	38	42	29,59			S	1	Cloudy.
	3	0	43	46	29,30			SW	2,3	Rain.
27	8	0	40	45	29,02			N	1	Cloudy.
	3	0	34	48	29,46			N	1,2	Fine.
28	8	0	33	44	29,91			S	1	Snow.
	3	0	42	47	29,84			W	1	Rain.
29	8	0	46	47	29,95			W	1	Cloudy.
	3	0	48	50	29,98			W	1	Cloudy.
30	8	0	42	48	30,15			W	1	Fine.
	3	0	44	50	30,39			NW	1	Fine.
31	8	0	33	46	30,54			W	1	Fine.
	3	0	39	47	30,52			W	1	Cloudy.

1815.	Thermometer without.			Thermometer within.			Barometer.*			Hygrometer.			Rain.†
	Greatest height.	Least height.	Mean height.	Greatest height.	Least height.	Mean height.	Greatest height.	Least height.	Mean height.	Greatest height.	Least height.	Mean height.	
	Deg.	Deg.	Deg.	Deg.	Deg.	Deg.	Inches.	Inches.	Inches.	Deg.	Deg.	Deg.	Inches.
January	45	22	34,0	50	39	45,0	30,47	28,94	29,55	78	62	71,7	0,435
February	54	36	43,6	58	48	54,2	30,51	29,38	29,81	79	63	73,3	0,675
March	63	34	47,6	63	48	55,1	30,30	28,90	29,75	79	58	70,2	1,325
April	63	40	49,4	67	50	55,6	30,60	28,78	29,66	79	55	67,0	1,660
May	68	50	58,2	69	55	60,4	30,33	29,62	29,74	75	56	65,4	0,667
June	70	52	61,6	70	57	63,2	30,29	29,30	29,84	74	50	61,6	1,752
July	72	55	62,9	72	54	65,0	30,21	29,62	30,03	67	48	57,8	1,581
August	69	52	63,5	69	52	63,0	30,27	29,56	29,94	70	51	63,9	0,222
September	70	54	64,7	70	54	63,6	30,20	29,49	29,97	69	56	61,8	0,776
October	63	44	53,2	63	49	59,0	30,29	29,33	30,33	—	—	—	1,889
November	54	29	41,2	58	44	50,5	30,61	29,05	29,97	—	—	—	1,336
December	48	26	38,7	54	41	47,8	30,54	28,95	29,79	—	—	—	0,650
Whole year			51,6			56,9			29,86				12,968

* The quicksilver in the bason of the barometer, is 81 feet above the level of low water spring tides at Somerset-house.

† The Society's Rain Gage is 114 feet above the same level, and 75 feet 6 inches above the surrounding ground.

Mean Variation of the Magnetic Needle, June, 1815, 24° 17′ 50″ West.

PHILOSOPHICAL

TRANSACTIONS,

OF THE

ROYAL SOCIETY

OF

LONDON.

FOR THE YEAR MDCCCXVI.

PART II.

LONDON,

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MDCCCXVI.

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PHILOSOPHICAL

TRANSACTIONS.

XI. *An essay towards the calculus of functions. Part* II. *By* C. Babbage, *Esq. Communicated by* W. H. Wollaston, *M. D. Sec. R. S.*

Read March 14, 1816.

In a former Paper which the Royal Society honoured with a place in the last Volume of their Transactions, I endeavoured to explain the nature of the calculus of functions, and I proposed means of solving a variety of functional equations containing only one variable quantity. My subsequent enquiries have produced several new methods of solving these, and much more complicated functional equations, and have convinced me of the importance of the calculus, particularly as an instrument of discovery in the more difficult branches of analysis; nor is it only in the recesses of this abstract science, that its advantages will be felt: it is peculiarly adapted to the discovery of those laws of action by which one particle of matter attracts or repels another of the same or of a different species; consequently, it may be applied to every branch of natural philosophy, where the object is to discover by calculation from the results of experiment, the laws which

regulate the action of the ultimate particles of bodies. To the accomplishment of these desirable purposes, it must be confessed that it is in its present state unequal; but should the labours of future enquirers give to it that perfection, which other methods of investigation have attained, it is not too much to hope, that its maturer age shall unveil the hidden laws which govern the phenomena of magnetic, electric, or even of chemical action.

When functional equations containing two or more variables occur, their solution presents still greater difficulties than those we have already considered; the new relations which arise, necessarily require a new notation to distinguish them. I shall endeavour, as far as I am able, to apply or extend that already in use; but, as it is almost impossible in the infancy of a calculus to foresee the extent to which it may be carried, or the new views which it may be necessary to take of it, the notation I have used should only be considered as of a temporary nature; it may be employed until some more convenient one be devised: perhaps, however, it might be more advantageous that it should not be altered until our acquaintance with this subject becomes more intimate, and until the infinitely varied and comprehensive relations displayed in the doctrine of functions, have been more minutely examined.

If ψ be the characteristic of any function of two quantities x and y, that function is thus denoted $\psi(x, y)$. Now, if instead of x in this quantity the original function be substituted, I shall call the result the second function relative to x, and I shall denote it thus

$$\psi^{2,1}(x, y) = \psi(\psi(x, y), y)$$

The first index 2 refers to x, and the second index 1 refers to y. Similarly if instead of y in the original function, the function itself had been substituted, the result would have been the second function relative to y; it would be thus denoted

$$\psi^{1,2}(x,y) = \psi(x,\psi(x,y))$$

If there are more than two variables in the original function, they may be arranged in the order in which they are to be operated on, and the indices will denote the number of operations to be performed.

Thus $\psi^{2,3,1,4}(x, y, z, v,)$ signifies that in the function $\psi(x,y,z,v,)$ we must instead of x substitute the function itself, and in the result instead of y put the same function, this latter operation must be repeated, and finally, instead of v in the last result, put the original function; this last operation must again be repeated twice.

There are many cases which this notation does not comprehend. If, for example, in the function just proposed, we wished again to take the function relative to x or y, it would not be easy to express this. The method I propose is to have two ranks of indices, the lower one to distinguish the quantities operated on; the upper one to mark the number of operations performed. According to this method the example just chosen would be written thus:

$$\overset{2,\,3,\,1,\,4}{\overset{1,\,2,\,3,\,4}{\psi}}(x,y,z,v)$$

If only such functions as these occur, we encumber our symbol without any advantage; if, however, we now wish to perform any farther operations, such, for instance, as to take the second function relative to z, and then the third relative to y,

we have a very convenient mode of doing it; these operations would be thus expressed;

$$\psi^{\substack{2,\,3,\,1,\,4,\,2,\,3\\1,\,2,\,3,\,4,\,3,\,2}}(x,y,z,v)$$

This notation may not appear sufficiently concise to those who do not consider the very complicated relation expressed by the above written symbol: it need, however, only be used in very few cases, and when the lower series of indices is omitted, it must always be understood, that the quantities themselves are arranged in the order in which they are to be operated on.

If in a function of two variable $\psi(x,y,)$ we take the second function relative to x, and then the second function relative to y, we have

$$\psi^{\substack{2,2\\1,2}}(x,y)=\psi\{\psi(x,\psi(x,y))\ \psi(x,y)\}$$

If we take the second function first relative to y, and then the second relative to x we shall find

$$\psi^{\substack{2,\,2\\2,\,1}}(x,y)=\psi\{\psi(x,y),\psi(\psi(x,y),y)\}$$

It appears from this, that the order in which these operations are performed is not immaterial, as the order in which we differentiate a function of two variables, is in the differential calculus.

The two expressions just given are the two second functions of $\psi(x,y)$, the first taken relative to x and y, and the second taken relative to y and x. But there may be another second function of $\psi(x,y)$, which will arise from substituting at the same time $\psi(x,y)$ for x, and $\psi(x,y)$ for y, it will be

$$\psi(\psi(x,y),\psi(x,y))$$

and may for the sake of distinction be called the second simultaneous function relative to x and y; it differs from the two preceding ones, and in order to denote it with brevity, I shall put a line over the two indices thus,

$$\psi^{\overline{2,\,2}}(x,y) = \psi\left(\psi\,(x,y),\,\psi\,(x,y)\right)$$

This method of distinguishing it is equally applicable when there are more variables.

There is only one other modification of the symbol denoting function to which I shall at present allude. Suppose (after any number of operations have been performed on a function of two variables for example) y becomes equal to x, and the result only is given: this will naturally be represented in a manner analogous to that in which Euler denoted the limits between which the integral of a quantity is to be taken.

Thus the equation $\psi^{\substack{2,\,2\\1,\,2}}(x,y) = fx \quad [y=x]$ arises from the following question: What is the form of a function of x and y, such that taking the second function relative to x, and then the second relative to y, the result on making y equal to x shall be a given function of x?

It might be proposed, that after putting y equal to x, the whole should be considered merely as a function of x, and that its n^{th} function should be taken on this hypothesis, and the result only should be given.

Such operations I would denote thus:

$$\psi^{\left.\substack{2,2\\1,2}\right\}^{n}}(x,y) = f(x)\ [y=x] \text{ or perhaps } \left\{\psi^{\substack{2,\,2\\1,\,2}}\right\}^{n}(x,y) = f(x)\,[y=x]$$

and in a similar manner all other relations of the same kind may be expressed.

I shall give one example which will illustrate these various modifications of the original functions,

$$\psi^{\left.\begin{matrix}2,\,3,\,\overline{2,\,2}\\3,\,1,\,2,\,4\end{matrix}\right\}\left.\begin{matrix}3,\,2\\3,\,1\end{matrix}\right\}3}(x, y, z, v) = f(x) \left[\begin{matrix}y = \alpha x\\ v = \beta z\end{matrix}\; z = \gamma x\right]$$

This equation contains the analytical enunciation of the following Problem.

What must be the form of a function of four quantities $\psi(x, y, z, v)$ such that taking the second function relative to z, the third relative to x, and the second simultaneous one relative to y and v: if in the result αx be put for y and βz for v, and the whole be then considered as a function of x and z, and if on this hypothesis the third function be taken relative to z, and the second relative to x; and if γx be now put for z and the third function of the expression considered merely as a function of x be taken, then it is required that the final result shall be equal to fx a given function of x?

Symmetrical functions I shall denote as in my former Paper, by putting a line over the quantities relative to which they are symmetrical, thus $\psi(\bar{x}, \bar{y}, \overset{1}{\bar{z}}, \overset{1}{\bar{v}})$ is symmetrical relative to x and y in one sense, and relative to z and v in another.

PROBLEM I.

Required the solution of the functional equation

$$\psi(x, y) = \psi(\alpha x, \beta y)$$

To avoid repetition α, β, γ, &c. unless otherwise mentioned, always express known functions, and φ, ψ, χ are unknown or arbitrary ones.

Put $\psi(x, y,) = \varphi(fx, f_1 y,)$

then the given equation becomes

$$\varphi\,(fx, f_1 y) = \varphi\,(f\alpha x, f_1 \beta y)$$

Determine f and f_1 from the two equations

$$fx = f\alpha x \text{ and } f_1 y = f_1 \beta y$$

this may be effected by Prob. I. or II. of my former Paper, take any particular solution and φ may remain perfectly arbitrary; then the general solution of the problem is

$$\psi\,(x, y) = \varphi\,(fx, f_1 y)$$

Ex. 1. Given the equation $\psi\,(x, y) = \psi\left(-x, \frac{1}{y}\right)$ here we have $f(x) = f(-x)$, and a particular solution is $fx = x^2$; also $f_1(y) = f_1\left(\frac{1}{y}\right)$ and a particular case is $f_1(y) = \frac{y^2+1}{y}$ hence the general solution of the equation is

$$\psi\,(x, y) = \varphi\left(x^2, \frac{y^2+1}{y}\right)$$

φ being perfectly arbitrary.

If we employ the general solutions of the equations $f(x) = f(-x)$ and $f_1(y) = f_1\left(\frac{1}{y}\right)$, we shall still only have one arbitrary function. In fact, the most general solution of the equation $\psi\,(x, y) = \psi\left(-x, \frac{1}{y}\right)$ with which I am at present acquainted is

$$\psi\,(x, y) = \varphi\left\{\overline{-x}, \overline{x}, \overset{1}{\overline{y}}, \overset{1}{\overline{\tfrac{1}{y}}}\right\}$$

and this only involves one arbitrary function.

PROBLEM II.

Given the same equation

$$\psi(x,y)=\psi(\alpha x,\beta y)$$

Suppose one particular solution of this equation is known, let it be $f(x,y)$,
then take $\psi(x,y)=\varphi f(x,y)$, φ being perfectly arbitrary and the given equation becomes

$$\varphi f(x,y)=\varphi f(\alpha x,\beta y)$$

which is evidently satisfied since $f(x,y)=f(\alpha x,\beta y)$ by the hypothesis.

Ex. 1. Let $\psi(x,y)=\psi\left(x^n,y^{\frac{1}{n}}\right)$
one particular solution of this equation is $f(x,y)=x^{\log. y}$
hence the general solution is

$$\psi(x,y)=\varphi\left(x^{\log. y}\right)$$

Ex. 2. Given the equation $\psi(x,y)=\psi(x^n,y^n)$ a particular case is $f(x,y)=\frac{\log. x}{\log. y}$, hence the general solution is

$$\psi(x,y)=\varphi\left(\frac{\log. x}{\log. y}\right)$$

Ex. 3. Given the equation $\psi(x,y)=\psi(x^n,y^m)$
In order to get a particular case let us put

$$f(x,y)=\log.^2 x+a\log.^2 y$$

by substituting this value we shall find that it is a particular solution of the equation, if $a=-\frac{\log. n}{\log. m}$,
hence the general solution of the equation is

$$\psi(x,y)=\varphi\left(\log.^2 x-\frac{\log. n}{\log. m}\log.^2 y\right)=\varphi\left(\log.\frac{\log. x}{(\log. y)^{\frac{\log. n}{\log. m}}}\right)$$

or changing the value of φ it becomes

$$\psi(x,y)=\varphi\left(\frac{(\log. x)^{\log. m}}{(\log. y)^{\log. n}}\right)$$

If $n = m$ we have $\psi(x,y) = \varphi\left(\frac{\log. x}{\log. y}\right)$ as in the last example, and if $m = \frac{1}{n}$, we have the same solution as in the first.

In these equations the functions have contained the variables separated; but it may frequently happen, that they occur mixed as in the following Problems.

PROBLEM III.

Given the equation

$$\psi(x,y) = \psi(\alpha(x,y), \beta(x,y))$$

Assume $\psi(x,y) = \varphi(f(x,y), f_{1}(x,y))$, and by making this substitution the equation becomes

$$\varphi\{f(x,y), f_{1}(x,y)\} = \varphi\{f(\alpha(x,y), \beta(x,y)), f_{1}(\alpha(x,y), \beta(x,y))\}$$

In order to render this equation identical, I determine f and f_{1} from the two following equations:

$$f(x,y) = f(\alpha(x,y), \beta(x,y)) \text{ and } f_{1}(x,y) = f_{1}(\alpha(x,y), \beta(x,y))$$

From these it appears, that f and f_{1} are merely two particular solutions of the original equation. If, therefore, we are acquainted with any, the general solution is

$$\psi(x,y) = \varphi(f(x,y), f_{1}(x,y))$$

If only one particular solution is known, the general one is

$$\psi(x,y) = \varphi f(x,y)$$

Ex. 1. Let us examine in what cases we can find the general solution of the equation

$$\psi(x,y) = \psi(x^{n} y^{m}, x^{k} y^{r})$$

In order to obtain a particular solution, put $\psi(x,y) = x^{v} y^{w}$ and making this substitution, we shall find the following equation of condition among the exponents.

$$(1-n)(1-r) = km$$

hence either of the following equations may be solved generally,

$$\psi(x,y) = \psi\left\{\frac{x^n}{y^{r-1}}, \frac{y^r}{x^{n-1}}\right\} \qquad (a)$$

$$\psi(x,y) = \psi\left\{y\left(\frac{x}{y}\right)^n, x\left(\frac{y}{x}\right)^r\right\} \qquad (b)$$

the solution of the former is $\psi(x,y) = \varphi(xy)$, and that of the latter is $\psi(x,y) = \varphi(x^{r-1}y^{n-1})$

In (a) put $n = r = \frac{1}{2}$, then the solution of $\psi(x, y) = (\sqrt{xy}, \sqrt{xy})$ is $\psi(x,y) = \varphi(xy)$

In a similar manner it may be found that the solution of

$$\psi(x,y) = \psi\left(\frac{x-y}{y}, \frac{x-y}{x}\right)$$

is
$$\psi(x,y) = \varphi\left(\frac{y}{x}\right)$$

As another example take the equation

$$\psi(x,y) = \psi\left(\frac{y}{2}\sqrt{\frac{y}{2x}}, \sqrt{2xy}\right)$$

a particular solution is $\psi(x,y) = 2xy + y^2$, hence the general solution is $\psi(x,y) = \varphi(2xy + y^2)$, but we may find another particular solution of this equation which is totally different from the former, and by combining the two we shall obtain a much more general solution. The equation (b) will coincide with the one under consideration, if we make $n = -\frac{1}{2}$ and $r = \frac{1}{2}$, then we shall have for another particular solution $\underset{1}{f}(x,y) = \frac{1}{y\sqrt{2xy}}$, hence a very general solution of the equation $\psi(x,y) = \psi\left(\frac{y}{2}\sqrt{\frac{y}{2x}}, \sqrt{2xy}\right)$ is

$$\psi(x,y) = \varphi\left\{2xy + y^2, \frac{1}{y\sqrt{2xy}}\right\}$$

φ remaining perfectly arbitrary.

In the equation of this Problem it may happen that

$\alpha(x,y)$ does not contain x nor $\beta(x,y)$ contain y, it then becomes $$\psi(x,y)=\psi(\alpha y,\beta x)$$

This is the case when $\psi(x,y)$ is required to be a symmetrical function of x and y, the equation would then become

$$\psi(x,y)=\psi(y,x)$$

two particular solutions are $f(x,y)=xy$ and $f_1(x,y)=x+y$, hence the general solution of the equation is

$$\psi(x,y)=\phi(xy,x+y)$$

Though these solutions may with propriety be termed general because they contain an arbitrary function, yet I am by no means inclined to think them the most general of which the questions admit, possibly we ought to except the two last equations, though I shall afterwards show that the solution of an equation of the form $\psi(x,y)=\psi(\alpha x,\beta y)$ may contain any number of known functions within the arbitrary one.

Problem IV.

Given the equation

$$\psi(x,y)=\psi(\alpha(x,y),\beta(x,y))$$

Assume as before $\psi(x,y)=\phi(f(x,y),f_1(x,y))$, then the equation will become

$$\phi(f(x,y),f_1(x,y))=\phi\left\{f(\alpha(x,y),\beta(x,y))f_1(\alpha(x,y),\beta(x,y)\right\}$$

In order to render this equation identical, determine f and f_1 from the two equations

$$f(x,y)=f_1(\alpha(x,y),\beta(x,y))\text{ and }f_1(x,y)=f(\alpha(x,y),\beta(x,y))$$

putting in the first of these $\alpha(x,y)$ for x and $\beta(x,y)$ for y we find

$$f(\alpha(x,y),\beta(x,y))=f_1\left\{\alpha(\alpha(x,y),\beta(x,y)),\beta(\alpha(x,y),\beta(x,y))\right\}$$
$$=f_1(x,y)\qquad(1)$$

and we should find a precisely similar equation for determining f_1. If we are acquainted with two particular solutions of this equation, we may from them derive the general solution of the given equation. If, however, the functions α and β are of such a nature that the two following equations are fulfilled eq. (1) becomes identical without assigning any particular value to f or f_1. (The two conditions are $\alpha(\alpha(x,y), \beta(x,y)) = x$ and $\beta(\alpha(x,y), \beta(x,y)) = y$).

It may be curious to enquire whether we can discover any forms which will satisfy these equations, for this purpose let us assume $\alpha(x,y) = a + bx + cy$, and also $\beta(x,y) = a + b_1 x + c_1 y$, this will only lead us to a particular solution, but I shall presently show that it may be rendered general. If the two conditions already specified are fulfilled, the arbitrary constants will be determined, and we shall have the following equations

$$\alpha(x,y) = a + bx + \frac{b^2-1}{b}y$$
$$\beta(x,y) = \frac{ab}{1-b} - bx - by$$

which may be thus generalised. Let φ be any function, and let φ^{-1} be the inverse of that function, so that $\varphi\varphi^{-1}x = x$ then the conditions will be fulfilled, if

$$\alpha(x,y) = \varphi^{-1}\left\{a + b\varphi x + \frac{b^2-1}{b}\varphi y\right\}$$

and $$\beta(x,y) = \varphi^{-1}\left\{\frac{ab}{1-b} - b\varphi x - b\varphi y\right\}$$

Some remarks, however, are necessary on the inverse function φ^{-1}. If we combine x and constant quantities by any of the direct operations, addition, multiplication, elevation of powers, &c. the result which is called a function of x admits

only of one value, let z equal the function, then we have the equation $z = \phi x$. If from this we endeavour to discover the value of x in terms of z, the operation is an inverse one and x admits of one or more values according to the nature of the operations denoted by ϕ. This number may even be infinite; if ϕ denotes an equation of the n^{th} degree, there are n values of x in terms of z. It may then be enquired whether in using the substitution employed in the latter part of this Problem, any of these (perhaps infinite number) may be taken, or whether only certain particular values should be used? without attending to this circumstance, our conclusions may become erroneous: all these different values will satisfy the equation $\phi\overset{-1}{\phi} x = x$, but only those must be used which also satisfy the equation $\phi\overset{-1}{\phi} x = x$: thus if $z = \phi x = a - x^2$ we shall have $x = \overset{-1}{\phi} z = \pm \sqrt{a - z}$ if we employ the upper sign we have

$$\phi\overset{-1}{\phi} x = + \sqrt{a - (a - x^2)} = + \sqrt{x^2} = + x$$

If we use the lower one

$$\overset{-1}{\phi}\phi x = - \sqrt{a - (a - x^2)} = - \sqrt{x^2} = - x$$

the upper sign must therefore be taken, because in the latter part of the Problem we suppose $\overset{-1}{\phi}\phi x = x$ and $\overset{-1}{\phi}\phi y = y$. This remark, which is of some importance, extends to the conclusions in my former Paper and to the whole of the subsequent enquiries.

The equation (1) might be considered as similar to the original one, and the same transformation might be performed on this, and thus we might continue to deduce new conditions. In the first part we found that the equation $\psi x = \psi \alpha x$ always admitted of an easy solution when $\alpha^n x = x$ and by continuing the substitutions already pointed out, we should arrive at

some conclusions very analogous for functional equations of the form of those treated of in this Problem, but the length to which these enquiries would lead, render it sufficient merely to indicate them.

In the equations solved in Problem I. and II. it is obviously immaterial whether we first put αx instead of x, and then in the result put βy for y or conversely; but in the equation of Problems III. and IV. the case is different. If in the function $\psi(x,y)$ we put simultaneously $\alpha\,(x,y)$ for x, and $\beta\,(x,y)$ for y the result will be different from that which would arise from first putting $\alpha\,(x,y)$ for x and then in the result putting $\beta\,(x,y)$ for y, or from inverting this operation; the three results stand thus:

$$\psi\,(\alpha\,(x,y), \beta\,(x,y)) \qquad (a)$$
$$\psi\,(\alpha\,(x, \beta\,(x,y)), \beta\,(x,y)) \qquad (b)$$
$$\psi\,(\alpha\,(x,y), \beta\,(\alpha\,(x,y),y)) \qquad (c)$$

These three functions are evidently different, and in the solutions of the Problems, regard was only had to the first of them which may be called the simultaneous function. Those, however, of the second and third class might occur, and it becomes necessary to point out the means of solution which are applicable to them.

According to the notation laid down, these functions may be thus expressed

$$\psi^{\overline{1,1}}_{1,1}(\alpha\,(x,y), \beta\,(x,y)) \qquad (a)$$
$$\psi^{1,2}_{1,1}(\alpha\,(x,y), \beta\,(x,y)) \qquad (b)$$
$$\psi^{2,1}(\alpha\,(x,y), \beta\,(x,y)) \qquad (c)$$

But to avoid the trouble of indices I shall show how those of

the second and third class may be reduced to those of the first, I shall therefore always consider functions of the first order as simultaneous ones, and omit the indices, which if supplied, would be $\substack{\overline{1,\,1,\,1}\\ 1,\,2,\,3}$, &c.

To transform $\psi^{\substack{1,\,1\\ 1,\,2}}(\alpha\, x,y), \beta\,(x,y))$ into a function whose index is $\substack{1,\,1\\ 1,\,2}$ put $\alpha\,(x, \beta\,(x,y)) = \gamma\,(x,y)$ then

$$\psi^{\substack{\overline{1,\,1}\\ 1,\,2}}(\gamma\,(x,y), \beta\,(x,y)) = \psi^{\substack{1,\,1\\ 1,\,2}}(\alpha\,(x,y), \beta(x,y))$$

and similarly if $\beta\,(\alpha\,(x,y), \beta\,(x,y)) = \gamma\,(x,y)$ we should have

$$\psi^{\substack{1,\,1\\ 2,\,1}}(\alpha\,(x,y), \beta\,(x,y)) = \psi^{\substack{\overline{1,\,1}\\ 1,\,2}}(\alpha\,(x,y), \gamma\,(x,y))$$

and generally whatever be the number of variables a similar transformation might be effected.

Problem V.

Required the solution of the equation.

$$\psi\,(x,y) = \mathrm{A}\,(x,y)\,\psi\,(\alpha\,(x,y), \beta\,(x,y))$$

Assume $\psi\,(x,y) = f\,(x,y)\,\varphi\,\left\{f_1(x,y), f_2(x,y)\right\}$ and substituting this in the given equation, we find

$$f(x,y)\,\varphi\left\{f_1(x,y), f_2(x,y)\right\} = \mathrm{A}\,(x,y) f\,(\alpha\,(x,y), \beta\,(x,y))$$
$$\times\,\varphi\left\{f_1(\alpha\,(x,y), \beta\,(x,y)), f_2(\alpha\,(x,y), \beta(x,y))\right\}$$

This equation will be satisfied if we are acquainted with particular solutions of the three following equations

$$f\,(x,y) = \mathrm{A}\,(x,y) f\,(\alpha\,(x,y), \beta\,(x,y))$$
$$f_1(x,y) = f_1(\alpha(x,y), \beta\,(x,y)) \text{ and } f_1(x,y) = f_1(\alpha\,(x,y), \beta\,(x,y))$$

the first of these is nothing more than the original equation.

If therefore we know one particular solution of the original equation, and also one or two particular solutions of the other equation, we may deduce the general solution of the Problem.

Ex. Let $\quad \psi(x,y) = \left(\frac{x}{y}\right)^3 \psi(y,x)$

in this case $f_1(x,y) = f_1(y,x)$ and two particular solutions are $f_1(x,y) = xy$ and $f_1(x,y) = x + y$ also a particular solution of the given equation is $f(x,y) = \frac{x}{y^2}$, hence its general solution is

$$\psi(x,y) = \frac{x}{y^2}\,\phi(x+y,\, xy)$$

PROBLEM VI.

Given the equation

$$\psi(x,y) = \mathrm{A}(x,y)\,\psi(\alpha(x,y), \beta(x,y)) + \mathrm{B}(x,y)$$

Suppose we are acquainted with one particular solution which satisfies the equation and let it be $f(x,y)$, then assume

$$\psi(x,y) = f(x,y) + \phi(x,y)$$

and making this substitution the equation becomes

$$f(x,y) + \phi(x,y) = \mathrm{A}(x,y) f(\alpha(x,y), \beta(x,y)) + \mathrm{A}(x,y) \times \phi(a(x,y), \beta(x,y)) + \mathrm{B}(x,y)$$

Substracting from this the particular solution

$$f(x,y) = \mathrm{A}(x,y) f(\alpha(x,y), \beta(x,y)) + \mathrm{B}(x,y)$$

there remains

$$\phi(x,y) = \mathrm{A}(x,y)\,\phi(\alpha(x,y), \beta(x,y))$$

an equation which may be solved by the preceding Problem. The same substitution is applicable to the more general equation

$$o = \psi(x,y) + \mathrm{A}(x,y)\,\psi(\alpha(x,y), \beta(x,y)) + \mathrm{B}(x,y)\,\psi(\alpha_1(x,y), \beta_1(x,y)) + \&c. + \mathrm{K}(x,y)$$

PROBLEM VII.

Given the functions

$$\alpha(x,y),\ \alpha_1(x,y),\ \alpha_2(x,y) \qquad \&c.$$

$$\beta(x,y),\ \beta_1(x,y),\ \beta_2(x,y) \qquad \&c.$$

Required the nature of the function $\psi(x,y)$ such that it shall not alter its form by the simultaneous substitution of $\alpha(x,y), \beta(x,y)$ for x and y, and generally that it shall remain the same when for x and y are respectively substituted any of the functions denoted by $\alpha_n(x,y)$ and $\beta_n(x,y)$. The conditions which determine ψ may be thus expressed

$$\psi(x,y) = \psi(\alpha(x,y), \beta(x,y)) = \psi(\alpha_1(x,y), \beta_1(x,y)) = \&c.$$

$$\text{Assume } \psi(x,y) = \varphi\left\{f(x,y), f_1(x,y)\right\} \qquad (1)$$

then from the first condition we have

$$\varphi\left\{f(x,y), f_1(x,y)\right\} = \varphi\left\{f(\alpha(x,y), \beta(x,y)), f_1(\alpha(x,y), \beta(x,y))\right\}$$

this will be satisfied by making

$$f(x,y) = f(\alpha(x,y), \beta(x,y)) \text{ and } f_1(x,y) = f_1(\alpha(x,y), \beta(x,y))$$

these are two particular solutions of the first equation.

The second condition is $\psi(x,y) = \psi(\alpha_1(x,y), \beta_1(x,y))$

which becomes

$$\varphi(f(x,y), f_1(x,y)) = \varphi(f(\alpha_1(x,y), \beta_1(x,y)), f_1(\alpha_1(x,y), \beta_1(x,y)) \quad (2)$$

where f and f_1 are known functions; make

$$f(\alpha_1(x,y), \beta_1(x,y)) = K(x,y) \text{ and } f_1(\alpha_1(x,y), \beta_1(v,y)) = {}'K(x,y)$$

K and 'K are therefore also known functions.

Assume $\phi(x,y) = \phi_1(f_2(x,y), f_3(x,y))$, then equation (2) becomes

$$\phi_1\{f_2(f(x,y), f_1(x,y)), f_3(f(x,y), f_1(x,y))\} = \phi_1\{f_2(\mathrm{K}(x,y), {}'\mathrm{K}(x,y)),$$
$$f_3(\mathrm{K}(x,y)\, {}'\mathrm{K}(x,y))\}$$

This equation must be solved in the same manner as the former by means of two particular solutions, and by continuing the same method, we shall find that the form of the function ψ may be determined by means of $2n$ particular solutions of certain functional equations, when there are n pair of conditions assigned. A less general solution may, however, be found when we are only acquainted with n particular solutions.

A similar method would lead us to the form of ψ, whatever might be the number of variables. If, however, we are acquainted with any number of particular solutions which remain the same, in all the cases assigned by the conditions of the Problem, we may have the general solution by making

$$\psi = \phi\{f_1, f_2, \ldots f_i\}$$

$f_1, f_2, \ldots f_i$ being it i particular solutions.

Ex. Let it be required to find a symmetrical function of $x_1, x_2, . x_n$, the equations to be satisfied are

$$\psi(x_1, x_2, \ldots x_n) = \psi(x_2, x_3, \ldots x_n, x_1) = \psi(x_3, x_4 \ldots x_n, x_1, x_2) = \&c.$$

or the whole of the conditions may be more concisely denoted by the expression

$$\psi\{\bar{x}_1, \bar{x}_2, \bar{x}_3 \ldots \bar{x}_n\}$$

We may easily find n particular solutions which fulfill these equations: for in the first place it is evident that the sum of

any number of quantities is symmetrical with respect to them, therefore

$$f_1(x_1, x_2, .. x_n) = x_1 + x_2 + \&c. + x_n = S(x_1)$$

Again the sum of their products two by two is also symmetrical, therefore

$$f_2(x_1, x_2, .. x_n) = x_1x_2 + x_2x_3 + x_1x_3 + \&c. = S(x_1x_2)$$

and similarly the sums of their products three by three, four by four, &c. are symmetrical. We may, therefore, find n different particular solutions, and the general solution will be any arbitrary function of all these particular solutions, or

$$\psi(\overline{x_1}, \overline{x_2}, .. \overline{x_n}) = \phi\left\{S(x_1), S(x_1x_2), \ldots S(x_1x_2 .. x_n)\right\}$$

Instead of taking for particular solutions the sum of all the quantities, the sum of all the products by two's, the sum of all the products by three's, &c. &c. we might have employed the sum of all the quantities, the sum of their squares, the sum of their cubes, &c. but the solution thus deduced would not be essentially different from the former.

On functional equations of the second and higher orders involving two or more variables.

The notation to be employed in these enquiries has already been sufficiently explained, and the different species of second functions have been noticed. Preserving the same symbols, let it be required to solve the following Problem.

PROBLEM VIII.

Given the equation

$$\psi^{2,1}(x, y) = x$$

This equation though apparently involving two variables

may in fact be solved by the methods of the first part; for y may be considered as a constant quantity, and if in the solution of $\varphi^2 x = x$ (Probs. 9, 10, and 14, Part I.) we put arbitrary functions of y instead of the constant quantities which occur, we shall have a solution of the given equation, thus a particular solution of $\varphi^2 x = x$ is $f(x) = \frac{b-x}{1-cx}$ instead of c put $\chi(y)$ then a solution of the given equation is

$$\psi(x, y) = \frac{b-x}{1-x\chi y}$$

for

$$\psi^{2,1}(x, y) = \psi(\psi x, y), y) \frac{b - \frac{b-x}{1-x\chi y}}{1 - \frac{b-x}{1-x\chi}\chi y} = \frac{x-bx\chi y}{1-b\chi y} = x$$

We might also instead of b put any other arbitrary function of y, and the result will be the same. The equations

$$\psi^{2,1}(x, y) = \alpha(x, y) \text{ and } \psi^{1,2}(x, y) = \alpha(x, y)$$

may be treated in a similar manner, in the first y must be considered as constant, and x must be so treated in the latter. In general, when functions are taken only relative to one of the variables, the rules delivered in my former Paper are sufficient for their solution, such is the equation

$$F\left\{x, y) \psi(x, y), \psi^{2,1}(x, y), \ldots \psi^{n,1}(x, y)\right\} = 0$$

It might however occur, that though the order of the function does not vary relative to the other variable, yet that that variable may occur in different forms in each function. An example will render this more evident α, β, &c. being known functions, let

$$F\left\{x, y, \psi(x, y), \psi^{2,1}(x, \alpha y), \psi^{3,1}(x, \beta y), \ldots \psi^{n,1}(x, \nu y)\right\} = 0$$

here though the functions do not vary in *order* relative to y,

yet they do vary in a certain sense, because y is differently contained under each functional characteristic; the method of treating these kind of equations will be explained hereafter.

PROBLEM IX.

Given the functional equation

$$\psi^{\overline{2,2}}(x,y)=o$$

This signifies that the second simultaneous function is equal to zero. It is evident that $x-y$ or $y-x$ will be a particular solution, for if $\psi(x,y)=x-y$ we have

$$\psi^{\overline{2,2}}(x,y)=\psi(\psi(x,y),\psi(x,y))=(x-y)-(x-y)=o$$

By observing the process just gone through, it appears that it would equally succeed if for x we put $f(x)$ and for y we put $f(y)$ for if $\psi(x,y)=fx-fy$, we have

$$\psi^{\overline{2,2}}(x,y)=(fx-fy)-(fx-fy)=o$$

This solution is considerably more general than the former, yet is by no means the complete solution, a more general one may be obtained thus: we found one particular solution to be $\psi(x,y)=x-y$, now if we multiply the right side of this equation by an arbitrary function of x and y the condition will still be fulfilled; for if $\psi(x,y)=(x-y)\,\phi(x,y)$ we shall find

$$\psi^{\overline{2,2}}(x,y)=\left\{\overline{x-y}\,\phi(x,y)-\overline{x-y}\,\phi(x,y)\right\}\times$$
$$\phi\left\{\overline{x-y}\,\phi(x,y),\overline{x-y}\,\phi(x,y)\right\}=o$$

provided $\phi\left\{\overline{x-y}\,\phi(x,y),\overline{x-y}\,\phi(x,y)\right\}$ does not contain in its denominator any factor which vanishes.

PROBLEM X.

Given the equation $\psi^{\overline{2,2}}(x,y)=a$

In this case the second simultaneous function of x and y is constant. The first solution which presents itself is $\psi(x,y)=x-y+A$, then we shall find

$$\psi^{\overline{2,2}}(x,y)=((x-y+A)-(x-y+A)+A=A=a$$

therefore $A=a$ and one particular solution is

$$\psi(x,y)=x-y+a$$

This may be rendered more general, nearly in the same manner as the last Problem; thus let $\psi(x,y)=(x-y)\,\phi(x,y)+a$ then

$$\psi^{\overline{2,2}}(x,y)=[(\overline{x-y}\,\phi(x,y)+a)-(\overline{x-y}\,\phi(x,y)+a)]\times \phi\left\{\overline{x-y}\,\phi(x,y)+a,\overline{x-y}\,\phi(x,y)+a\right\}+a=a$$

Another particular solution which readily occurs is

$$\psi(x,y)=A\frac{x}{y} \text{ this gives } \psi^{\overline{2,2}}(x,y)=A\frac{A\frac{x}{y}}{A\frac{x}{y}}=A=a$$

therefore $A=a$ and a particular solution is

$$\psi(x,y)=\frac{ax}{y} \text{ or } \psi(x,y)=\frac{ay}{x}$$

this readily points out another general solution, let

$$\psi(x,y)=A\phi\left(\frac{x}{y}\right) \text{ hence } \psi^{\overline{2,2}}(x,y)=A\phi\left(\frac{A\phi\frac{x}{y}}{A\phi\frac{x}{y}}\right)=A\phi(1)=a$$

make $A=\frac{a}{\phi(1)}$ and the general solution is

$$\psi(x,y)=\frac{a}{\phi(1)}\phi\left(\frac{x}{y}\right)$$

From the combination of the two preceding solutions we

may obtain another value of ψ which will also satisfy the given equation; it will be

$$\psi(x,y) = \frac{a\chi\left\{\overline{x-y}\,\varphi(x,y),\ \underset{1}{\varphi}\frac{x}{y}\right\}}{\chi\left\{0,\ \underset{1}{\varphi}(1)\right\}}$$

This on trial will be found to agree with the condition, and χ, φ and $\underset{1}{\varphi}$ are arbitrary functions

The equation we are considering will also be satisfied by making $\psi(x,y) = a\frac{\varphi(x)}{\varphi(y)}$ or more generally by the constant quantity a multiplied by any fraction whose numerator and denominator become equal when x is put for y: such are the following.

$$a\frac{x+xy+y^2}{y+2x^2},\ a\frac{x(y^2+x^2)}{2y^3},\ a\frac{\sqrt{x^3+8x^2y-5y^3}}{2x\sqrt{x}},\ \&c.$$

Problem XI.

Given the equation

$$\psi^{\overline{n,n}}(x,y) = ax + by$$

Assume $\psi(x,y) = px + qy$ then we have

$$\psi^{\overline{2,2}}(x,y) = p(px+qy) + q(px+qy) = (p+q)(px+qy)$$

and $$\psi^{\overline{3,3}}(x,y) = (p+q)^2(px+qy)$$

and generally

$$\psi^{n,n}(x,y) = (p+q)^{m-1}(px+qy)$$

hence $p.(p+q)^{n-1} = a$ and $q(p+q)^{n-1} = b$, which gives for the values of p and q

$$p = \frac{a}{(a+b)^{\frac{n-1}{n}}} \text{ and } q = \frac{b}{(a+b)^{\frac{n-1}{n}}}$$

This is a very limited solution not containing even an arbitrary constant, it might easily be rendered more general, but

the problem itself would scarcely have been worth noticing had it not been for the very curious results to which it led me.

The relation between $\psi(x, y) = px + qy$ and $\psi^{\overline{n,n}}(x, y) = (p+q)^{n-1}(px+qy)$ is very remarkable, it appears from this, that in the present case in going from the n^{th} to the $n+1^{th}$ simultaneous function, we have only to multiply by the sum of the co-efficients of the original function. On enquiring a little more minutely into the cause of this circumstance, it will be found that it depends on the original function containing x and y of the same dimensions in all its terms, or more generally that the expression of $\psi(x, y)$ is homogeneous. Let us now assume some homogeneous function, and examine its second and higher simultaneous functions, let

$$\psi(x, y) = ax^n + by^p x^{n-p} + cy^q x^{n-q} + \&c.$$

the second simultaneous function is

$$\psi^{\overline{2,2}}(x, y) = a\{\psi(x, y)\}^n + b\{\psi(x, y)\}^n + c\{\psi(x\ y)\}^n + \&c.$$

or $\psi^{\overline{2,2}}(x, y) = \{\psi(x, y)\}^n\{a+b+c+\&c.\} = \psi(1,1)\{\psi(x, y)\}^n (a)$

If we now take the simultaneous third functions we have

$$\psi^{\overline{3,3}}(x, y) = \psi(1,1)[\psi^{2,2}(x, y)\ ^n = \psi(1,1)[\psi(1,1)(\psi(x, y))^n]^n$$

hence $\quad \psi^{\overline{3,3}}(x, y) = \{\psi(1,1)\}^{1+n}\{\psi(x, y)\}^{n^2}$

Repeating the same operation we should have

$$\psi^{\overline{4,4}}(x, y) = \{\psi(1,1)\}^{1+n+n^2} \times \{\psi(x, y)\}^{n^3}$$

and generally

$$\psi^{\overline{k,k}}(x, y) = \{\psi(x, y)\}^{n^{k-1}} \times \{\psi(1,1)\}^{1+n+\&c.+n^{k-2}} =$$

$$= \{\psi(x, y)\}^{n^{k-1}} \times \{\psi(1,1)\}^{\frac{1-n^{k-1}}{1-n}} \quad (b)$$

This elegant property of homogeneous functions will assist us in solving a variety of equations.

PROBLEM XII.

Given the equation

$$\psi^{\overline{n,n}}(x,y)=a\{\psi(x,y)\}^{b}$$

Determine n from the equation $b=n^{k-1}$ and also determine $\psi(1,1)$ from the equation $\{\psi(1,1)\}^{\frac{1-n^{k-1}}{1-n}}=a$

Or the given equation will be satisfied by any homogeneous function of the degree indicated by $b^{\frac{1}{k-1}}$ provided the sum of all its coefficients is equal to the quantity $a.^{\frac{1-b^{\frac{1}{k-1}}}{1-b}}$

Ex. Let $\psi^{\overline{3,3}}(x,y)=8(\psi(x,y))^4$

here $b=4, k=3$, therefore $n^{k-1}=n^2=b=4$ and $n=\pm 2$

also $a=8$ and $\psi(1,1)=8^{\frac{1}{3}}=2$

therefore any of the following quantities will satisfy the equation $2xy$, x^2+y^2, $xy+y^2$ $x^2-xy+2y^2$

The properties of homogeneous functions are so nearly connected with the solution of equations containing simultaneous functions, that it will be convenient to examine into them a little farther, and to adopt some means of denoting them with brevity. In order to signify that a function of several variables has in each of its terms the sum of the indices of any two of them always the same, I shall make use of a line placed beneath those two variables: thus $\psi(\underline{x,y})$ signifies an homogeneous function of x and y; and

as it may be convenient also to denote the sum of the two indices, I shall place it underneath on the outside of the parenthesis; thus then the expression $\psi(\underline{x}, \underline{y})_q$ denotes any homogeneous function of x and y of the q^{th} degree. A function of three variables x, y, and z, may be homogeneous, with respect to two of them (x and y) in one sense, and also relative to y and z in another; but it does not from thence follow that it will be homogeneous relative to all three, such a function would be denoted thus

$$\psi(\underline{x}, \underline{\underline{y}}, \underline{z})_{p,\,q}$$

a particular case of this expression is $x^2 z^2 + xyz = \psi(\underline{x}, \underline{\underline{y}}, \underline{z})_{2,\,2}$

This notation being premised, we have the following theorems relative to homogeneous functions.

$$\psi^{\overline{2,\,2}}(\underline{x}, \underline{y})_n = \psi(1, 1)\,[\psi(\underline{x}, \underline{y})_n]^n \qquad (1)$$

$$\psi^{\overline{k,\,k}}(\underline{x}, \underline{y})_n = \left\{\psi(\underline{x}, \underline{y})_n\right\}^{n^{k-1}} \times \left\{\psi(1, 1)^{\frac{1-n^{k-1}}{1-n}} \qquad (2)$$

And generally if we have any homogeneous function of the n^{th} degree, and instead of x and y we substitute any other function whatever as $\underset{1}{\psi}(x, y)$, then we shall have the following equation

$$\psi\left\{\underline{\underset{1}{\psi}(x, y)}, \underline{\underset{1}{\psi}(x, y)}\right\}_n = \psi(1, 1) \times [\psi(x, y)]^k \qquad (3)$$

$$\text{Assume } \psi(x, y) = \phi\left\{\frac{\alpha(\underline{x}, \underline{y})_n}{\beta(\underline{x}, \underline{y})_m}\right\}$$

call the latter member, for the sake of brevity, K, and take the second simultaneous function on both sides; in this case $\alpha(\underline{x}, \underline{y})_n$ will become $\alpha(1, 1)\,K^n$ by eq. (3), and for the same reason $\beta(\underline{x}, \underline{y})_m$ will become $\beta(1, 1)\,K^m$, and consequently we shall have

$$\psi^{\overline{2,2}}(x,y)=\phi\left\{\frac{\alpha(1,1)}{\beta(1,1)}K^{n-m}\right\}=\phi\left\{\frac{\alpha(1,1)}{\beta(1,1)}\left(\phi\,\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)^{n-m}\right\} \quad (4)$$

let $\alpha(1,1)=\beta(1,1)$ and also let $m=n$, then this equation becomes

$$\psi^{\overline{2,2}}(x,y)=\phi(1)$$

this affords another and a more direct solution of Prob. 10. for if $\psi^{\overline{2,2}}(x,y)=a$, a general solution is

$$\psi(x,y)=\frac{a}{\phi(1)}\phi\left\{\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_n}\right\}$$

$\alpha(1,1)$ being equal to $\beta(1,1)$, this latter condition, however, is not absolutely necessary, and if we wish to avoid it, the general solution will be

$$\psi(x,y)=\frac{a}{\phi\left(\frac{\alpha(1,1)}{\beta(1,1)}\right)}\phi\left\{\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_n}\right\}$$

the following is another solution of the same question

Assume

$$\psi(x,y)=\phi\left\{\frac{\begin{matrix}a+bx+cx^2+\&c.\\ \phantom{a+{}}\underset{1}{b}y+\underset{1}{c}xy\\ \underset{2}{c}y^2\end{matrix}}{\begin{matrix}{}^1a+{}^1bx+{}^1cx^2+\&c.\\ {}^1\underset{1}{b}y+{}^1\underset{1}{c}xy\\ {}^1\underset{2}{c}y^2\end{matrix}}\right\}=K$$

taking the second function on both sides we have

$$\psi^{\overline{2,2}}(x,y)=\phi\left\{\frac{\left.\begin{matrix}a+b\\ \underset{1}{b}\end{matrix}\right\}K+\left.\begin{matrix}c\\ \underset{1}{c}\\ \underset{2}{c}\end{matrix}\right\}K^2+\&c.}{\left.\begin{matrix}{}^1a+{}^1b\\ {}^1\underset{1}{b}\end{matrix}\right\}K+\left.\begin{matrix}{}^1c\\ {}^1\underset{1}{c}\\ {}^1\underset{2}{c}\end{matrix}\right\}K^2+\&c.}\right\}$$

let $a={}^1a \quad b+\underset{1}{b}={}^1b+{}^1\underset{1}{b} \quad c+\underset{1}{c}+\underset{2}{c}={}^1c+{}^1\underset{1}{c}+{}^1\underset{2}{c}$ &c.

then the equation is reduced to

$$\psi^{\overline{2,2}}(x,y)=\phi(1)$$

If therefore we assume

$$\psi(x,y)=\frac{a}{\phi(1)}\,\phi\left\{\frac{\begin{array}{llll} a+bx & +cx^2 & + \&c. \\ \quad by & \;\; c\underset{1}{xy} & \\ & \;\; \underset{2}{cy^2} & \end{array}}{\begin{array}{llll} {}^1a+{}^1bx & +{}^1cx^2 & + \&c. \\ \quad \underset{1}{{}^1by} & \;\; \underset{1}{{}^1cxy} & \\ & \;\; \underset{2}{{}^1cy^2} & \end{array}}\right\}$$

the original equation will be satisfied.

I am inclined to think, that this solution is not the most general of which the Problem admits, even though the series were continued back, as it might, to negative powers of x and y. The two solutions which follow are possibly more general, although on this point I am not certain. It would indeed be a very important step, if we could assign the number and nature of the arbitrary functions which enter into the complete solution of functional equations.

Another solution of the equation $\psi^{\overline{2,2}}(x,y)=a$ may be thus deduced, let

$$\psi(x,y)=\phi\left\{\frac{\alpha(\underline{x,y})_m}{\underset{1}{\alpha}(\underline{x,y})_m},\ \frac{\beta(\underline{x,y})_n}{\underset{1}{\beta}(\underline{x,y})_n},\ \&c.\right\}=K$$

then taking the second simultaneous function on both sides, it will be perceived by the construction of the second side of the equation, that

$$\psi^{\overline{2,2}}(x,y)=\phi\left\{\frac{\alpha(1,1)}{\underset{1}{\alpha}(1,1)},\ \frac{\beta(1,1)}{\underset{1}{\beta}(1,1)},\ \&c.\right\}$$

call the right side of the equation A, then a very general

solution of our equation is

$$\psi(x, y) = \frac{a}{A}\,\phi\left\{\frac{a(x, y)_m}{\underset{1}{\alpha}(x, y)_m}, \frac{\beta(x, y)_n}{\underset{1}{\beta}(x, y)_n}, \&c.\right\}$$

where the numbers m, n, p, &c. are not confined to integers. Another solution may be found in the following manner:

let $$\psi(x, y) = \phi\left(\frac{\chi(x, y)}{\underset{1}{\chi}(x, y)}\right) = K$$

and determine $\underset{1}{\chi}$ so that $\underset{1}{\chi}(x, y) = \chi(x, y)$ when y is made equal to x, then taking the second simultaneous functions on both sides, we have

$$\psi^{\overline{2, 2}}(x, y) = \phi\left(\frac{\chi(K, K)}{\underset{1}{\chi}(K, K)}\right) = \phi(1)$$

a general solution of the equation in question is therefore

$$\psi(x, y) = \frac{a}{\phi(1)}\,\phi\left(\frac{\chi(x, y)}{\underset{1}{\chi}(x, y)}\right)$$

this solution depends on that of the equation

$$\underset{1}{\chi}(x, y) = \chi(x, y) \qquad [y = x]$$

which belongs to a class of equations we shall speak of hereafter.

Let us now return to the consideration of equation (4) it is

$$\psi^{\overline{2, 2}}(x,y) = \phi\left\{\frac{\alpha(1, 1)}{\beta(1, 1)}\left[\phi\left(\frac{\alpha(x, y)_n}{\beta(x, y)_m}\right)\right]^{n-m}\right\}$$

for n put $n+1$ and for m put n, then it becomes

$$\psi^{\overline{2, 2}}(x, y) = \phi\left\{\frac{\alpha(1, 1)}{\beta(1, 1)}\phi\left(\frac{\alpha(x, y)_{n+1}}{\beta(x, y)_n}\right)\right\}$$

take the third simultaneous function then

$$\psi^{\overline{3, 3}}(x, y) = \phi\left\{\frac{\alpha(1, 1)}{\beta(1, 1)}\phi\left(\frac{\alpha(1, 1)}{\beta(1, 1)}\phi\left(\frac{\alpha(x, y)_{n+1}}{\beta(x, y)_n}\right)\right)\right\} \qquad (5)$$

if we suppose $\alpha(1,1) = \beta(1,1)$ these equations become

$$\psi^{\overline{2,2}}(x,y) = \phi^2\left(\frac{\alpha(\underline{x,y})_{n+1}}{\beta(\underline{x,y})_n}\right)$$

$$\psi^{\overline{3,3}}(x,y) = \phi^3\left(\frac{\alpha(\underline{x,y})_{n+1}}{\beta(\underline{x,y})_n}\right)$$

and generally we should find

$$\psi^{\overline{p,p}}(x,y) = \phi^p\left(\frac{\alpha(\underline{x,y})_{n+1}}{\beta(\underline{x,y})_n}\right) \qquad (6)$$

where $\psi(x,y) = \phi\left(\frac{\alpha(\underline{x,y})_{n+1}}{\beta(\underline{x,y})_n}\right)$ and also $\alpha(1,1) = \beta(1,1)$

A more general expression, and one which contains (b) as a particular case may be deduced in the following manner.

$$\text{Let} \qquad \psi(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\,\phi\,\frac{\alpha(\underline{x,y})_n}{\beta(\underline{x,y})_m}\right\}^{\frac{1}{n-m}} = \mathrm{K}$$

taking the second simultaneous functions on both sides we find

$$\psi^{\overline{2,2}}(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\,\phi\,\frac{\alpha(1,1)\,\mathrm{K}^n}{\beta(1,1)\,\mathrm{K}^m}\right\}^{\frac{1}{n-m}} = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\,\phi\,\frac{\alpha(1,1)}{\beta(1,1)}\,\mathrm{K}^{n-m}\right\}^{\frac{1}{n-m}} =$$

$$= \left\{\frac{\beta\,1,1)}{\alpha(1,1)}\,\phi\left(\frac{\alpha(1,1)}{\beta(1,1)}\frac{\beta(1,1)}{\alpha(1,1)}\,\phi\,\frac{\alpha(\underline{x,y})_n}{\beta(\underline{x,y})_m}\right)\right\}^{\frac{1}{n\ m}} = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\,\phi^2\left(\frac{\alpha(\underline{x,y})_n}{\beta(\underline{x,y})_m}\right)\right\}^{\frac{1}{n-m}}$$

and if we continue to take the succeeding simultaneous functions we shall find generally, that when

$$\psi(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\,\phi\left(\frac{\alpha(\underline{x,y})_n}{\beta(\underline{x,y})_m}\right)\right\}^{\frac{1}{n-m}}$$

$$\psi^{\overline{p,p}}(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\,\phi^p\left(\frac{\alpha(\underline{x,y})_n}{\beta(\underline{x,y})_m}\right)\right\}^{\frac{1}{n-m}} \qquad (7)$$

this expression is much more general than the preceding (b)

with which it coincides when $\alpha\,(1,\,1) = \beta\,(1,\,1)$ and $n = m + 1$ by their assistance we may solve a variety of problems relating to simultaneous functions.

From (b) we have

$$\psi^{\overline{2,\,2}}(x,y) = \varphi^2\left(\frac{\alpha\,(\underline{x,y})_{n+1}}{\beta\,(\underline{x,y})_n}\right) = \varphi\,\varphi\left(\frac{\alpha\,(\underline{x,y})_{n+1}}{\beta\,(\underline{x,y})_n}\right)$$

putting in this for $\varphi\left(\frac{\alpha\,(\underline{x,y})_{n+1}}{\beta\,(\underline{x,y})_n}\right)$ its value $\psi\,(x,\,y)$ we have

$$\psi^{\overline{2,\,2}}(x,\,y) = \varphi\,\psi\,(x,y)$$

from this we may deduce the solution of the following Problem.

PROBLEM XIII.

Given the equation

$$\psi^{\overline{2,\,2}}(x,\,y) = \mathrm{F}\,\psi\,(x,y)$$

make $\varphi = \mathrm{F}$ and take $\alpha\,(\underline{x,y})_{n+1}$ any homogeneous function of the $\overline{n+1}th$ degree, and $\beta\,(\underline{x,y})_n$ a similar function of the n^{th}, also let $\alpha\,(1,\,1) = \beta\,(1,\,1)$ then the equation is satisfied by making

$$\psi\,(x,\,y) = \mathrm{F}\left\{\frac{\alpha\,(\underline{x,y})_{n+1}}{\beta\,(\underline{x,y})_n}\right\}$$

Ex. Let $\psi^{\overline{2,\,2}}(x,\,y) = \sqrt{\psi\,(x,y)}$

Suppose $\alpha\,(\underline{x,\,y})_{n+1} = x^2 + y^2$ and $\beta\,(x,y)_n = 2x$, then one solution is $\psi\,(x,y) = \sqrt{\frac{x^2+y^2}{2x}}$

or let $\alpha\,(\underline{x,\,y})_{n+1} = (x^2 + y^2)\,\varphi\,(1)$ and $\beta\,(\underline{x,\,y})_n = 2\,y\,\varphi\left(\frac{x}{y}\right)$

then a more general solution is $\psi\,(x,y) = \sqrt{\frac{x^2+y^2}{2\,y\,\varphi\,\frac{x}{y}}\,\varphi\,(1)}$

PROBLEM XIV.

Given the equation

$$\psi^{\overline{n,n}}(x,y)=F\psi(x,y)$$

In equation (6) we have

$$\psi^{\overline{n,n}}(x,y)=\varphi^{n}\left(\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x,y)}_{n}}\right)=\varphi^{n-1}\varphi\left(\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x,y)}_{n}}\right)$$

put for $\varphi\left(\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x,y)}_{n}}\right)$ its value $\psi(x,y)$ then we have

$$\psi^{\overline{n,n}}(x,y)=\varphi^{n-1}\psi(x,y)=F\psi(x,y)$$

determine φ from the equation $\varphi^{n-1}v=Fv$ by Prob. 13. Part 1. and the general solution of the equation is

$$\psi(x,y)=\varphi\left(\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x,y)}_{n}}\right)$$

Ex. 1. Let $\psi^{\overline{3,3}}(x,y)=\psi(x,y)$

in this case $\varphi^{n-1}v=Fv$ becomes $\varphi^{2}v=v$ solutions of which are

$$\varphi v=\frac{a}{v},\quad \varphi v=a-v,\quad \varphi v,=(a-v^{m})^{\frac{1}{m}}$$

let $\alpha\,\underline{(x,y)}_{n+1}=x^2+y^2$ and $\beta\,\underline{(x,y)}_{n}=x+y$ then solutions of the equation $\psi^{\overline{3,3}}(x,y)=\psi(x,y)$ are

$$\psi(x,y)=\frac{a(x+y)}{x^2+y^2},\quad \psi(x,y)=a-\frac{x^2+y^2}{x+y}$$

$$\text{and } \psi(x,y)=\left\{a-\left(\frac{x^2+y^2}{x+y}\right)^{m}\right\}^{\frac{1}{m}}$$

Ex. 2. Let $\psi^{\overline{n,n}}(x,y)=\left\{\psi(x,y)\right\}^{m}$

in this example $\varphi^{n-1}v=Fv$ becomes $\varphi^{n-1}v=v^{m}$ and $\varphi v=v^{m^{\frac{1}{n-1}}}$

its particular solutions are therefore

$$\left(\frac{x^2+y^2}{x+y}\right)^{m^{\frac{1}{n-1}}}\qquad \left(\frac{x^2+2xy-y^2}{y^2}\right)^{m^{\frac{1}{n-1}}}$$

other more general ones are

$$\psi(x,y)=\left\{\frac{(x^2+y^2)\,\varphi(1)}{2\,x\,\varphi\left(\frac{x}{y}\right)}\right\}^{m\frac{1}{n-1}} \text{ and } \psi(x,y)=\left\{\frac{2\,x\,y\,\varphi(1)}{(x+y)\,\varphi\left(\frac{y}{x}\right)}\right\}^{m\frac{1}{n-1}}$$

PROBLEM XV.

Required the solution of the equation

$$\psi^{\overline{n,\,n}}(x,y)=\mathrm{F}\,\psi^{\overline{n-p,\,n-p}}(x,y)$$

In (6) we have

$$\psi^{\overline{n,\,n}}(x,y)=\varphi^{n}\left(\frac{\alpha\,\underline{(x,y)}_{n-1}}{\beta\,\underline{(x,y)}_{n}}\right)=\varphi^{p}\,\varphi^{n-p}\left(\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x,y)}_{n}}\right)$$

but $\psi\,(x,y)=\varphi\left(\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x,y)}_{n}}\right)$ consequently $\varphi^{n-p}\left(\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x,y)}_{n}}\right)=$ $\psi^{\overline{n-p,\,n-p}}(x,y)$ and substituting this value, we shall find

$$\psi^{\overline{n,\,n}}(x,y)=\varphi^{p}\,\psi^{\overline{n-p,\,n-p}}(x,y)$$

make $\varphi^{p}\,v=\mathrm{F}v$ and find the value φ, then the general solution of the equation is $\psi\,(x,y)=\varphi\left(\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x,y)}_{n}}\right)$ if $\alpha\,(1,1)=\beta\,(1,1)$.

Ex. 1. Let $\psi^{\overline{4,\,4}}(x,y)=\left\{\psi^{\overline{2,\,2}}(x,y)\right\}^{2}$

here $\varphi^{2}v=\mathrm{F}v=v^{2}$ and $\varphi\,v=v^{\sqrt{2}}$

therefore $\psi\,(x,y)=\left\{\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x,y)}_{n}}\right\}^{\sqrt{2}}$ if $\alpha\,(1,1)=\beta\,(1,1)$

more particular solutions are

$$\psi\,(x,y)=\left(\frac{x+y}{2\,\varphi\,\frac{y}{x}}\varphi\,(1)\right)^{\sqrt{2}} \text{ and } \psi\,(x,y)=\left(\frac{x^2+xy+y^2}{3y}\right)^{\sqrt{2}}$$

Problem XVI.

Given the equation

$$F\left\{\psi(x,y),\ \psi^{\overline{2,2}}(x,y),\ \ldots \&c.\ \psi^{\overline{p,p}}(x,y)\right\}=0$$

The same substitution as that employed in the last Problem will reduce this equation of two variables to a similar one which contains only one

putting $\psi(x,y)=\varphi\left(\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x,y)}_n}\right)$ we have $\psi^{\overline{2,2}}(x,y)=\varphi^2\left(\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x,y)}_n}\right)$

and $\psi^{\overline{p,p}}(x,y)=\varphi^{p}\left(\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x.y)}_n}\right)$ also making $\frac{\alpha\,\underline{(x,y)}_{n+1}}{\beta\,\underline{(x,y)}_n}=v$

the given equation becomes

$$F\left\{\varphi v,\ \varphi^2 v,\ \ldots\ \varphi^p v\right\}=0$$

an equation which contains only one variable, and may therefore be solved by the methods described in the first part.

Problem XVII.

Required the solution of the equation

$$F\left\{\psi(x,y),\ \psi^{\overline{2,2}}(x,y),\ \ldots\ \psi^{\overline{p,p}}(x,y)\right\}=0$$

The following considerations lead to another mode of solution applicable to this Problem. If in the function $\psi(x,y)$ we put y equal to x it becomes $\psi(x,x)$ call this φx: then if in the second simultaneous function of $\psi(x,y)$ we put y equal x, the result will be the same as if we had taken the second function of $\psi(x,x)$ or φx relative to x, or symbollically expressed, it is

$$\psi^{\overline{2,2}}(x,y)=\psi\big(\psi(x,x),\ \psi(x,x)\big),=\varphi^2 x \qquad [y=x]$$

this may be rendered evident by substituting for the right side of the equation its value $\psi(\psi(x,y),\ \psi(x,y))$.

In the same manner it may be shown, that if we take the p^{th} simultaneous function, and then put x for y, the result will be the same as the p^{th} function of φx, or expressed in symbols it is

$$\psi^{\overline{p,p}}(x,y)=\varphi^{p}(x) \quad [y=x]$$

Now since this equation is identical when y is equal to x, it will remain so when any other quantity as v is put for x, if the same quantity is also put for y, therefore

$$\psi^{\overline{p,p}}(v,v)=\varphi^{p}(v)$$

now let $v=\psi(x,y)$ this equation becomes

$$\psi^{\overline{p,p}}(\psi(x,y),\psi(x,y))=\varphi^{p}\psi(x,y)$$

but the right side of this equation is nothing more than the $\overline{p+1}th$ simultaneous function of $\psi(x,y)$, consequently

$$\psi^{\overline{p+1},\overline{p+1}}(x,y)=\psi^{p}\psi(x,y)$$

If now in the equation of the Problem we substitute the several values thus formed of the simultaneous functions, we shall have

$$F\left\{\psi(x,y),\varphi\psi(x,y),\varphi^{2}\psi(x,y),\ldots\varphi^{p-1}\psi(x,y)\right\}=0$$

and putting z for $\psi(x,y)$ we have

$$F\left\{z,\varphi z,\varphi^{2}z,\ldots\varphi^{p-1}z\right\}=0$$

which is a functional equation of one variable, and may be solved by the methods of the first Part. The form of φ being thus ascertained, we have for determining $\psi(x,y)$ the equation

$$\psi(x,y)=\varphi x \quad [y=x]$$

or expressed in words $\psi(x,y)$ may be any function of x and y which becomes equal to φx when y is equal to x.

PROBLEM XVIII.

Given the equation $\psi^{\overline{p,\,p}}(x,y) = \mathrm{F}(x,y)$
$\mathrm{F}(x,y)$ being such a function of x and y that it may be reducible to the form $\underset{1}{\mathrm{F}}\,\gamma(\underline{x},\underline{y})$.

Assume $\psi(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\,\phi\left(\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)\right\}^{\frac{1}{n-m}}$

then from eq. (7) we have

$$\psi^{\overline{p,\,p}}(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\,\phi^p\left(\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)\right\}^{\frac{1}{n-m}}$$

make $\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m} = \gamma(\underline{x},\underline{y}) = v$, and since $\gamma(\underline{x},\underline{y})$ is given and α and β are indeterminate, this equation may be easily satisfied in an infinite number of ways; put v for $\gamma(\underline{x},\underline{y})$ our equation becomes

$$\left\{\frac{\beta(1,1)}{\alpha(1,1)}\,\phi^p\,v\right\}^{\frac{1}{n-m}} \underset{1}{=} \mathrm{F}v$$

which contains only one variable and may be solved by the methods of the former part.

Ex. 1. Let $\psi^{\overline{p,\,p}}(x,y) = \frac{x+y}{2f(1)}f\left(\frac{x}{y}\right)$

Assume $\alpha(\underline{x},\underline{y}) = (x+y)f\left(\frac{x}{y}\right)$ and $\beta(\underline{x},\underline{y}) = 2f(1)$

then it becomes

$$\phi^p\,v = v$$

and calling $\underset{1}{f}$ any particular solution of this equation we have for the general one

$$\psi(x,y) = \overset{-1}{\phi}\,\underset{1}{f}\,\phi\left(\frac{x+y}{2f(1)}\cdot f\left(\frac{x}{y}\right)\right)$$

Ex. 2. Let $\psi^{\overline{3,3}}(x,y) = \frac{x^2}{y}$
put $\alpha(\underline{x},\underline{y}) = x^2$ and $\beta(\underline{x},\underline{y}) = y$
then the equation to be solved is $\varphi^3 v = v$ a particular solution of which is $\varphi v = \frac{1+v}{1-3v}$ putting for v its value and using the the general solution we have

$$\psi(x,y) = \overset{-1}{\varphi}\left\{\frac{1+\varphi\frac{x^2}{y}}{1-3\varphi\frac{x^2}{y}}\right\}$$

as a particular case, take

$$\psi(x,y) = \left\{\frac{y^n + x^{2n}}{y^n - 3x^{2n}}\right\}^{\frac{1}{n}}$$

which will be found on trial to satisfy the condition.

Problem XIX.

Given the equation

$$F\left\{\gamma(\underline{x},\underline{y}), \psi(x,y), \psi^{\overline{2,2}}(x,y), \ldots \psi^{\overline{p,p}}(x,y)\right\} = 0$$

This equation is evidently capable of solution by the same means as the last; putting

$$\psi(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\,\varphi\left(\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)\right\}^{\frac{1}{n-m}}$$

we have as before

$$\psi^{\overline{p,p}}(x,y) = \left\{\frac{\beta(1,1)}{\alpha(1,1)}\,\varphi^p\left(\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m}\right)\right\}^{\frac{1}{n-m}}$$

and assuming α and β such that $\frac{\alpha(\underline{x},\underline{y})_n}{\beta(\underline{x},\underline{y})_m} = \gamma(\underline{x},\underline{y}) = v$ and

making $\frac{\beta(1,1)}{\alpha(1,1)} = a$ our equation becomes

$$F\left\{ v, (a\phi v)^{\frac{1}{n-m}}, (a\phi^2 v)^{\frac{1}{n-m}}, \ldots (a\phi^p)^{\frac{1}{n-m}} \right\} = 0$$

which may be solved by Prob. XIX. Part I.

Problem XX.

Given the equation

$$\psi^{2,1}(x,y) = \psi^{1,2}(x,y)$$

This equation, containing no simultaneous function, is different from any we have yet solved, and requires the application of a peculiar artifice.

In my former Paper, in order to reduce the equation $\psi^2 x = \alpha x$ to one of the first order, I made use of the substitution $\overset{-1}{\phi} f \phi x$ for ψx: an analogous one must be employed on the present occasion; let us suppose

$$\psi(x,y) = \overset{-1}{\phi} f(\phi x, \phi y)$$

the effect of this will be very similar to that of the one just alluded to, and its great utility will be evident by considering its result in the various orders of the same function, thus

$$\psi^{2,1}(x,y) = \overset{-1}{\phi} f(\phi \overset{-1}{\phi} f(\phi x, \phi y), \phi y) = \overset{-1}{\phi} f(f(\phi x, \phi y), \phi y)$$
$$= \overset{-1}{\phi} f^{2,1}(\phi x, \phi y)$$

$$\psi^{1,2}(x,y) = \overset{-1}{\phi} f(\phi x, \phi \overset{-1}{\phi} f(\phi x, \phi y)) = \overset{-1}{\phi} f(\phi x, f(\phi x, \phi y))$$
$$= \overset{-1}{\phi} f^{1,2}(\phi x, \phi y)$$

$$\psi^{\overline{2,2}}(x,y) = \overset{-1}{\phi} f(\phi \overset{-1}{\phi} f(\phi x, \phi y), \phi \overset{-1}{\phi} f(\phi x, \phi y)) = \overset{-1}{\phi} f(f(\phi x, \phi y),$$
$$f(\phi x, \phi y)) = \overset{-1}{\phi} f^{\overline{2,2}}(\phi x, \phi y)$$

and continuing the same substitutions we shall find

$\psi^{3,1}(x,y) = \overset{-1}{\phi} f^{3,1}(\phi x, \phi y)$ and $\psi^{1,3}(x,y) = \overset{-1}{\phi} f^{1,3}(\phi x, \phi y)$

and generally

$$\psi^{n,m}(x,y) = \overset{-1}{\phi} f^{n,m}(\phi x, \phi y)$$

If there are more variables than two, the proper substitution to be made is

$$\psi(x_1, x_2, \ldots x_i) = \overset{-1}{\phi} f^{1,1,\&c.}(\phi x_1, \phi x_2, \ldots \phi x_i)$$

and there would result generally

$$\psi^{n,m,p,\&c}(x_1, x_2, \ldots x_i) = \overset{-1}{\phi} f^{n,m,p,\&c.}(\phi x_1, \phi x_2, \ldots \phi x_i)$$

By such substitutions all simple functional equations of every order and of any number of variables, may be reduced to those of the first order: but the difficulty is not then overcome, the resulting equations are by no means easy to solve, and in a variety of cases it appears, that they are contradictory or impossible.

Let us apply this substitution to the equation of this Problem, then since $\psi(x,y) = \overset{-1}{\phi} f(\phi x, \phi y)$ we have

$$\overset{-1}{\phi} f^{2,1}(\phi x, \phi y) = \overset{-1}{\phi} f^{1,2}(\phi x, \phi y)$$

Performing the operation denoted by $\overset{-1}{\phi}$ on both sides it becomes $\qquad f^{2,1}(\phi x, \phi y) = f^{2,1}(\phi x, \phi y)$

Put $\overset{-1}{\phi} x$ for x and $\overset{-1}{\phi} y$ for y then it becomes

$$f^{2,1}(x,y) = f^{1,2}(x,y)$$

which is nothing more than the original equation; from it, however we learn, that if we can find one particular solution, we can always deduce from it the general one, which supposing f a particular case, will be

$$\psi(x,y) = \overset{-1}{\phi} f(\phi x, \phi y)$$

After repeated endeavours I have been unable to find any particular case which will satisfy the equation

$$\psi^{2,1}(x,y) = \psi^{1,2}(x,y)$$

I have also made some attempts at discovering particular solutions of the two following equations, and have met with no success.

$$\psi^{\overline{2,2}}(x,y) = \psi^{1,2}(x,y) \text{ and } \psi^{\overline{2,2}}(x,y) = \psi^{2,1}(x,y)$$

Should, however, any particular case be found, their general solutions flow immediately from the method just explained.

With regard to the equation of the Problem $\psi^{2,1}(x,y) = \psi^{1,2}(x,y)$, I have little expectation of finding any particular case, as I think the following reasoning, though perhaps not quite so satisfactory as might be wished, will show the impossibility of it. First, let us suppose, that $\psi(x,y)$ is a symmetrical function of x and y, let it be $\chi(\overline{x},\overline{y})$ then our equation becomes

$$\chi\left\{\overline{\chi(\overline{x},\overline{y}),\overline{y}}\right\} = \chi\left\{\overline{x},\overline{\chi(\overline{x},\overline{y})}\right\} = \chi\left\{\overline{\chi(\overline{x},\overline{y}),\overline{x}}\right\}$$

Comparing the first of these expressions with the third, we may observe that in the first, wherever $\chi(\overline{x},\overline{y})$ occurs, the same quantity $\chi(\overline{x},\overline{y})$ also occurs in the third, consequently in this respect, the first and third are identical: but wherever y occurs in the first, x occurs similarly in the third, therefore in this respect they cannot be identical, unless y is equal to x. From this it appears, that the equation in question cannot be solved by any symmetrical function. Again, the given equation $\psi^{2,1}(x,y) = \psi^{1,2}(x,y)$ contains x and y in the same manner, and no reason can be assigned why in the solution x should be contained differently from y: this may, perhaps, be made more clear, thus. Let $f(x,y)$ be the quantity to which each side of the given equation is equal, then

$$\psi^{2,1}(x,y) = f(x,y) = \psi^{1,2}(x,y)$$

Now since $\psi(\psi(x,y),y) = f(x,y)$ and also $\psi(x,\psi(x,y)) =$

$f(x,y)$; and since taking the second functions is a direct operation, it is evident that the original function $\psi\,(x,\,y)$ will produce the same result, whether we take the second function relative to x or relative to y; therefore it must be similarly composed of x and y; that is to say, it must be symmetrical relative to x and y: but we have before shown that no symmetrical function can satisfy the equation, consequently the equation is contradictory.

This train of reasoning I offer with considerable hesitation, well aware of the extreme difficulty of reasoning correctly on a subject so very general, and which, from its novelty, the mind has not been sufficiently habituated to consider, so as to rely with confidence on any lengthened process of reasoning. I thought it, however, right to mention this proof, that those who may seek for particular cases, might first enquire whether the equation be possible.

Problem XXI.

Given the equation

$$x\,\psi^{1,2}(x,y) = y\,\psi^{2,1}(x,y)$$

Substituting $\overset{-1}{\varphi} f\,(\varphi x,\,\varphi y)$ for $\psi\,(x,y)$ in this equation we have

$$x\,\overset{-1}{\varphi} f^{1,2}(\varphi x,\varphi y) = y\,\overset{-1}{\varphi} f^{2,1}(\varphi x,\,\varphi y)$$

putting $\overset{-1}{\varphi}\,x$ for x and $\overset{-1}{\varphi}\,y$ for y it becomes

$$\overset{-1}{\varphi}\,x\;.\;\overset{-1}{\varphi} f^{1,2}(x,y) = \overset{-1}{\varphi}\,y\;.\;\overset{-1}{\varphi} f^{2,1}(x,y)$$

This equation will be satisfied if we could find such a form for f, that the two following equations might be fulfilled.

$$f^{1,2}(x,y) = y \text{ and } f^{2,1}(x,y) = x$$

for in that case it would become

$$\overset{-1}{\varphi}\,x.\,\overset{-1}{\varphi}\,y = \overset{-1}{\varphi}\,y\;.\;\overset{-1}{\varphi}\,x$$

which is identical.

Our enquiries must therefore be directed to this point, and it will be found that $f(x,y)=a-x-y$ has the required properties, and is a particular solution of the given equation: hence the general solution is

$$\psi(x,y)=\overset{-1}{\phi}(a-\phi x-\phi y)$$

There are many other particular cases which fulfil the same condition, such as

$$f(x,y)=\frac{a}{xy} \text{ and } f(x,y)=\frac{1-x-y}{1-bxy}$$

these give the general solutions

$$\psi(x,y)=\overset{-1}{\phi}\frac{a}{\phi x\,\phi y} \text{ and } \psi(x,y)=\overset{-1}{\phi}\left(\frac{1-\phi x-\phi y}{1-b\phi x\,\phi y}\right)$$

PROBLEM XXII.

Given the equation

$$\psi^{2,1}(x,y)\,.\,\psi^{1,2}(x,y)=xy$$

Using the same substitution employed in the last Problem, this equation becomes

$$\overset{-1}{\phi}f^{2,1}(\phi x,\phi y)\,.\,\overset{-1}{\phi}f^{2,1}(\phi x,\phi y)=xy$$

and putting $\overset{-1}{\phi}x$ for x, and $\overset{-1}{\phi}y$ for y we have

$$\overset{-1}{\phi}f^{2,1}(x,y)\,.\,\overset{-1}{\phi}f^{1,2}(x,y)=\overset{-1}{\phi}x\,.\,\overset{-1}{\phi}y$$

which becomes identical, if $f^{2,1}(x,y)=x$ and $f^{1,2}(x,y)=y$ consequently all the solutions of the last Problem also solve this.

PROBLEM XXIII.

Given the equation

$$F\left\{x,\psi^{1,2}(x,y)\right\}=F\left\{\psi^{2,1}(x,y),y\right\}$$

this equation may be solved by the same artifice as the two last, assuming $\psi(x,y)=\overset{-1}{\phi}f(\phi x,\phi y)$ we have

$$F\left\{x,\overset{-1}{\phi}f^{1,2}(\phi x,\phi y)\right\}=F\left\{\overset{-1}{\phi}f^{2,1}(\phi x,\phi y),y\right\}$$

and putting $\overline{\phi}^{1} x$ for x, and $\overline{\phi}^{1} y$ for y, it becomes

$$F\left\{\overline{\phi}^{1} x, \overline{\phi}^{1} f^{1,2}(x,y)\right\} = F\left\{\overline{\phi}^{1} f^{2,1}(x,y), \overline{\phi}^{1} y\right\}$$

this is identical if we assume f so that the two conditions $f^{1,2}(x,y) = y$ and $f^{2,1}(x,y) = x$ may be fulfilled.

The same method is applicable to the equation

$$\left\{\psi^{2,1}(x,y) - x\right\} F(x, y, \psi(x,y) \text{ \&c.}) = \left\{\psi^{1,2}(x,y) - y\right\} \underset{1}{F}(x, y, \psi(x,y), \text{ \&c.})$$

for the two factors which multiply F and $\underset{1}{F}$ vanish on account of the value of f.

Problem XXIV.

Given the equation

$$x \psi^{\overline{2,2}}(x,y) = a \psi^{2,1}(x,y)$$

this equation, by means of the substitution already so frequently employed, becomes

$$x \overline{\phi}^{1} f^{\overline{2,2}}(\phi x, \phi y) = a \overline{\phi}^{1} f^{2,1}(\phi x, \phi y)$$

and putting $\overline{\phi}^{1} x$ for x, and $\overline{\phi}^{1} y$ for y we have

$$\overline{\phi}^{1} x . \overline{\phi}^{1} f^{\overline{2,2}}(x,y) = a \overline{\phi}^{1} f^{2,1}(x,y)$$

An artifice somewhat similar to the one already employed, will afford the solution of this equation: if we can find such a value of $f(x, y,)$ that $f^{\overline{2,2}}(x,y) = c$ and also $f^{2,1}(x,y) = x$ the equation will become identical by making $c = \phi a$. Such a value of f is $f(x,y) = c \frac{y}{x}$ for

$$f^{2,2}(x,y) = c \frac{c \frac{y}{x}}{c \frac{y}{x}} = c \text{ and } f^{2,1}(x,y) = c \frac{\frac{y}{x}}{c \frac{y}{x}} = x$$

hence the general solution of the given equation is

$$\psi(x,y) = \overline{\phi}^{1}\left(\frac{\phi a \phi y}{\phi x}\right)$$

PROBLEM XXV.

Given the equation

$$F\left\{\psi^{\overline{2,2}}(x,y),\, x\right\} = F\left\{a,\, \psi^{2,1}(x,y)\right\}$$

making the substitution $\overset{-1}{\phi} f(\phi x, \phi y)$ for $\psi(x,y)$ and in the result putting $\overset{-1}{\phi} x$ for x, and $\overset{-1}{\phi} y$ for y, we shall find

$$F\left\{\overset{-1}{\phi} f^{\overline{2,2}}(x,y),\, \overset{-1}{\phi} x\right\} = F\left\{a,\, \overset{-1}{\phi} f^{2,1}(x,y)\right\}$$

which will become identical if we select such a value for $f(x,y)$ that $\overset{-1}{\phi} f^{\overline{2,2}}(x,y) = a$, and also $f^{2,1}(x,y) = x$ such a value is $f(x,y) = \phi a + y - x$; hence the general solution of the Problem is

$$\psi(x,y) = \overset{-1}{\phi}(\phi a + \phi y - \phi x)$$

the more general equation

$$\left\{\psi^{2,1}(x,y) - x\right\} F\left\{x, y, \psi(x,y)\ \&c.\right\} = \psi^{\overline{2,2}}(x,y)\,.\, \underset{1}{F}\left\{x, y, \psi(x,y), \&c.\right\}$$

may be solved nearly in the same manner, its solution will be

$$\psi(x,y) = \overset{-1}{\phi}(\phi y - \phi x)$$

PROBLEM XXVI.

Given the equation

$$F\left\{x, y, \psi(x,y), \psi^{1,2}(x,y), \psi^{2,1}(x,y), \&c.\right\} = o$$

Assume $\psi(x,y) = \overset{-1}{\phi} f(\phi x, \phi y)$ then we have $\psi^{2,1}(x,y) = \overset{-1}{\phi} f^{2,1}(\phi x, \phi y)$, and $\psi^{1,2}(x,y) = \overset{-1}{\phi} f^{1,2}(x,y)$ and generally $\psi^{n,m}(x,y) = \overset{-1}{\phi} f^{n,m}(\phi x, \phi y)$

Substituting these values the equation becomes

$$F\left\{x, y, \overset{-1}{\phi} f(\phi x, \phi y), \overset{-1}{\phi} f^{1,2}(\phi x, \phi y), \overset{-1}{\phi} f^{2,1}(\phi x, \phi y), \&c.\right\} = o$$

and putting $\overset{-1}{\phi}x$ for x and $\overset{-1}{\phi}y$ for y we have

$$F\left\{\overset{-1}{\phi}x, \overset{-1}{\phi}y, \overset{-1}{\phi}f(x,y), \overset{-1}{\phi}f^{1,2}(x,y), \overset{-1}{\phi}f^{2,1}(x,y), \&c.\right\}=0$$

Some particular value of f must now be assumed, and the equation treated as one of the first order relative to $\overset{-1}{\phi}$.

The form to be assigned to f is of some consequence; it ought to be a particular solution of the original equation: for if we assign to it any other form, this adds a limitation to the original equation which may or may not agree with it, a particular solution should therefore always be employed. This remark is applicable to several Problems in my former Paper, and with this restriction, their solutions will remain correct.

Problem XXVII.

To transform the equation

$$F\left\{x, y, \psi(x,y), \psi^{2,1}(\alpha x, \beta y), \psi^{1,2}(\alpha_1 x, \beta_1 y), \&c.\right\}=0$$

into the form of the equation of the preceding Problem.

Assume $\psi(x,y) = \overset{-1}{\phi}f(\phi x, \phi y)$, then the equation becomes

$$F\left\{x, y, \overset{-1}{\phi}f(\phi x,\phi y), \overset{-1}{\phi}f^{2,1}(\phi\alpha x, \phi\beta y), \overset{-1}{\phi}f^{1,2}(\phi\alpha_1 x, \phi\beta_1 y), \&c.\right\}=0$$

find for ϕ by Prob. VII. Part I. such a value that it shall not change when any of the following quantities are substituted for x.

$$\begin{array}{cc} \alpha x & \beta x \\ \alpha_1 x & \beta_1 x \\ \alpha_2 x & \beta_2 x \\ \&c. & \&c. \end{array}$$

let this value be A, then the equation becomes

$$F\left\{x, y, \overset{-1}{A}f(Ax, Ay,) \overset{-1}{A}f^{2,1}(Ax, Ay), \&c.\right\} = 0$$

or putting $\bar{A}^{-1}x$ for x, and $\bar{A}^{-1}y$ for y we have

$$F\left\{\bar{A}^{-1}x,\ \bar{A}^{-1}y,\ \bar{A}^{-1}f(x,y),\ \bar{A}^{-1}f^{2,1}(x,y),\ \bar{A}^{-1}f^{1,2}(x,y),\ \&c.\right\}=o$$

an equation of the required form.

One important use of this transformation is the solution of equations of the form

$$F\left\{x,y,\ \psi(x,y),\ \psi^{1,2}(\alpha x,y),\ \psi^{1,2}\beta x,y),\ \&c.\right\}=o$$

in which the functions are taken only relative to y, and yet x is not altogether constant. It may be transformed into

$$\underset{1}{F}\left\{x,y,\psi(x,y),\psi^{1,2}(x,y),\psi^{1,3}(x,y),\ \&c.\right\}=o$$

which may be treated as an equation of one variable, x being constant. This is the species of equation alluded to at page (198)

After considering the various equations amongst the higher orders of functions, another question presents itself, which may be thus stated. What must be the form of a function of (n) variables, such that taking the functions relative to any or to all of them any number of times, and combining these quantities in any manner, the result shall (when all these variables are made equal to x) be equal to a given function of x? This question might thus be expressed when there are only two variables

$$F\left\{x,y,\psi(x,y),\psi^{1,2}(x,y),\psi^{2,1}(x,y),\ \&c.\right\}=f(a)\quad[y=x]$$

this condition obviously enlarges the signification of the function ψ, and the solutions ought to be more general. We shall accordingly find that some equations, of which without this condition we cannot find even a particular solution, are capable, when it is added, of very extensive ones. When there are more than two variables, the condition may be, that making

them equal by pairs, the result shall be given: a particular case would be the equation

$$F\left\{x, y, z, v, w, r, \psi(x, y, z, v, w, r, \&c.\right\} = F_{1}\left\{x, v, r\right\} \begin{bmatrix} y = x \\ z = v \\ w = r \end{bmatrix}$$

Instead of making y equal to x, and z to v and so on, y might become a given function of x, and z a given function of v, &c. thus:

$$F\left\{x, y, z, v, w, r, \psi(x, y, z, v, w, r,), \&c.\right\} = F_{1}\left\{x, v, r,\right\} \begin{bmatrix} y = \alpha x \\ z = \beta v \\ w = \gamma r \end{bmatrix}$$

a few examples will sufficiently explain the method to be pursued in treating these equations.

Problem XXVIII.

Given the equation

$$\psi^{2,1}(x, y) = \psi^{1,2}(x, y) \qquad [y = x]$$

This Problem, of which without the condition of y being made equal to x we could not find even a particular case, readily admits of solution in its present state. Since $\psi^{2,1}(x, y)$ is only equal to $\psi^{1,2}(x, y)$, when $y = x$ we may put for the given equation

$$\psi^{2,1}(x, y) = \phi\left\{(x, y), \psi^{1,2}(x, y)\right\} \qquad (1)$$

provided that when y is equal to x, the latter side of the equation shall become $\psi^{1,2}(x, y)$, and this fully satisfies the condition of the Problem. If, therefore, we can find such a value of ϕ, the equation (1) may be treated as a common functional equation of two variables, and may be solved by the rules already given.

Nor is it at all difficult to find such a value of ϕ; if we make $\psi^{1,2}(x, y) = z$, ϕ must be such a function that

$$\phi(x, y, z) = z \qquad [y = x]$$

It is evident that particular values of φ are

$$x - y + z \quad \text{and} \; \frac{x}{y} z$$

many others might be mentioned, but it is desirable to determine φ more generally

Since $\qquad \varphi(x, y, z) = z \quad [y = x]$

it is evident that $\quad \varphi(x, x, z) = z$

and since this is independent on any particular value of x we have

$$\varphi(v, v, z) = z$$

that is to say, that whatever quantity is substituted for x, if the same quantity is also substituted for y, the result will be equal to z. Now let $v = \varphi(x, y, z)$, it becomes

$$\varphi\left\{\varphi(x, y, z), \varphi(x, y, z), z\right\} = z$$

but this expression is nothing more than the second simultaneous function relative to x and y, and may be therefore more concisely expressed thus

$$\varphi^{\overline{2, 2}, 1}(x, y, z) = z$$

in which equation, since it does not vary relative to z, that quantity may be considered as a constant; and the equation

$$\varphi^{\overline{2, 2}}(x, y) = z = \text{constant}$$

being solved, we have only to substitute instead of the various constant quantities arbitrary functions of z: thus then the solution of the equation

$$\varphi(x, y, z) = x \quad [y = x]$$

is reduced to that of

$$\psi^{\overline{2, 2}}(x, y) = \text{constant}$$

and we have only to refer to Problem (10) for its general solution.

Let us apply this to the solution of the equation of this Problem $\qquad \psi^{2, 1}(x, y) = \psi^{1, 2}(x, y) \quad [y = x]$

take as a particular case of the equation $\overline{\phi^{2,2,1}}(x,y,z)=z$ $\phi(x,y,z)=\frac{x}{y}z$, then the equation of the Problem becomes

$$\psi^{2,1}(x,y)=\frac{x}{y}\psi^{1,2}(x,y) \text{ or } \qquad y\psi^{2,1}(x,y)=x\psi^{1,2}(x,y)$$

this is the equation solved in Prob. XXI. therefore all its solutions are also solutions of this equation. This however is, comparatively speaking, but a very limited answer: every different solution of the equation $\overline{\phi^{2,2,1}}(x,y,z)=z$ furnishes a new solution of our Problem, containing one or more arbitrary functions; each of these may very justly be called a general solution; but to investigate the number and nature of the arbitrary constants which enter into the complete solution, is an enquiry of considerable difficulty.

Problem XXIX.

Given the equation

$$\psi^{\left.\begin{smallmatrix}2,\,2\\1,\,2\end{smallmatrix}\right\}n}(x,y)=\text{F}(x) \qquad [y=\alpha x]$$

This signifies, that after taking the second function relative to x, and then the second relative to y; the result is considered merely as a function of x, and its n^{th} function taken relative to that variable: lastly, the quantity to which this becomes equal, after performing these operations, is given. The manner of treating these equations is very simple; put

$\psi^{2,2}(x,x)=\chi(x)$, then our equation becomes

$$\psi^{\left.\begin{smallmatrix}2,\,2\\1,\,2\end{smallmatrix}\right\}n}(x,y)=\chi^n(x)=\text{F}(x) \qquad [y=\alpha x]$$

determine χ from the equation $\chi^n x=\text{F}(x)$ by Prob. XIII. Part I. and let its solution be $\underset{1}{\text{F}}(x)$, then we have

$$\psi^{2,2}(x,y)=\underset{1}{\text{F}}(x) \qquad [y=\alpha x]$$

This equation may be solved by nearly the same method as that employed in the last Problem.

If the function occurs in different shapes or of various orders, this method is inapplicable, as in the following Problem.

PROBLEM XXX.

Given the equation

$$F\left\{\psi^{2,1)2}(x,y),\ \psi^{1,2)3}(x,y),\ x,\ y\right\}=0 \qquad [y=x]$$

The difficulties in this case appear to be much encreased from this circumstance, that the second function of $\psi^{2,1}(x,x)$ relative to x is quite different from the second function of $\psi^{1,2}(x,x)$ relative to the same quantity. The method of solution which I shall explain is equally applicable to all of this species, and consists in reducing them to a class which has been already solved.

It may be observed, that whether we take the second function of $\psi^{2,1}(x,x)$ relative to x, or whether we take the simultaneous function of $\psi^{2,1}(x,y)$ considered as a simple function, and in the result put x for y, the two expressions will be the same; the first gives

$$\psi^{2,1}(\psi^{2,1}(x,x),\ \psi^{2,1}(x,x))$$

and the second is

$$\psi^{2,1}(\psi^{2,1}(x,y),\psi^{2,1}(x,y))$$

which when y becomes equal to x is identical with the former; but

$$\psi^{2,1}(\psi^{2,1}(x,y),\psi^{2,1}(x,y))=\psi^{2,1}\left\{\psi(\psi(x,y),y),\psi(\psi(x,y),y)\right\}$$

$$=\psi^{\substack{2,\,1,\,\overline{1,\,1},\,1\\1\ \ 2\ \ 1\ \ 2\ \ 1}}(x,y)$$

the lower line of indices denoting the quantities relative to which the operations are performed. In a similar manner it may be shown, that

$$\psi^{\substack{1,\,2)3\\ }}(x,y)=\psi^{\substack{1,\,2,\,\overline{2,\,2},\,2\\1,\,2,\,1,\,2,\,2}}(x,y)$$

Substituting these values in the original equation, we have

$$F\left\{\psi^{\substack{2,\,1,\,\overline{1,\,1},\,1\\1,\,2,\,1,\,2,\,1}}(x,y),\ \psi^{\substack{1,\,2,\,\overline{2,\,2},\,2\\1,\,2,\,1,\,2,\,2}}(x,y),\ x,\ y\right\}=0 \quad [y=x]$$

This is an equation similar to that of Problem XXVIII., and may be solved by the same means.

New methods of solving functional equations of the first order, and also differential functional equations.

The new methods which I now propose to explain are only applicable to equations of the form

$$F\left\{x,\ \psi x,\ \psi\alpha x,\ \psi\alpha^2 x,\ \ldots\ \psi\alpha^n x\right\}=0$$

where α must be such a function that $\alpha^{n+1}x=x$. By the method of Prob. VII. Part I. all functional equations of the first order may be reduced to this form; and although in many cases this reduction is very difficult, or even in the present state of analysis out of our power, yet it is theoretically possible, and we shall therefore consider all equations as so reduced. There is this remarkable difference between the former methods and the present one:

Those which I have already given always led to the general solution, and perhaps, in some cases, to the complete one; these, on the contrary, which I shall now propose, always conduct us directly to a particular solution, which does not contain even an arbitrary constant. It has, however, several advantages; it is the most direct method with which we are yet acquainted; and if by any means we could introduce into these solutions an arbitrary constant, it would afford us general ones: this is a step which is wanting to connect it with the former methods. In the case of differential functional

equations, this step is supplied by the integrations which are necessary, and we thus arrive at their general solutions.

PROBLEM XXXI.

Given the equation

$$F\left\{x,\ \psi x,\ \psi \alpha x\right\} = 0$$

and also $\alpha^2 x = x$

Find ψx in terms of x and $\psi \alpha x$; let it be

$$\psi x = \underset{1}{F}\left\{x,\ \psi \alpha x\right\}$$

put αx for x; then it becomes

$$\psi \alpha x = \underset{1}{F}\left\{\alpha x,\ \psi \alpha^2 x\right\} = F\left\{\alpha x,\ \psi x\right\}$$

put this value of $\psi \alpha x$ in the former equation, and we have

$$\psi x = \underset{1}{F}\left\{x,\ \underset{1}{F}\left(\alpha x,\ \psi x\right)\right\}$$

from which equation ψx may be found in terms of x.

If $\alpha^3 x = x$ instead of $\alpha^2 x = x$, we should find

$$\psi x = \underset{1}{F}\left\{x,\ \underset{1}{F}\left\{\alpha x,\ \underset{1}{F}\left\{\alpha^2 x,\ \psi x\right\}\right\}\right\}$$

Ex. 1. Take the equation $(\psi x)^p \cdot \psi(a-x) = x^n$ where $\alpha x = a - x$

then $$\psi x = \left(\frac{x^n}{\psi(a-x)}\right)^{\frac{1}{p}}$$

and $$\psi(a-x) = \left\{\frac{(a-x)^n}{\psi x}\right\}^{\frac{1}{p}}$$

this substituted in the former gives

$$\psi x = \left\{\frac{x^n}{\left(\frac{(a-x)^n}{\psi x}\right)^{\frac{1}{p}}}\right\}^{\frac{1}{p}}$$

from which we find

$$\psi x = \left\{\frac{x}{(a-x)^{\frac{1}{p}}}\right\}^{\frac{np}{p^2-1}}$$

which will be found on trial to satisfy the equation.

Ex. 2. Given the equation $x^n \psi x - a\psi \frac{1}{x} = bx^p$

here $\alpha x = \frac{1}{x}$ and $\alpha^2 x = x$, and we have

$$\psi x = \frac{bx^p - a\psi \frac{1}{x}}{x^n}$$

and putting $\frac{1}{x}$ for x it becomes

$$\psi \frac{1}{x} = \frac{b\overline{x}^p - a\psi x}{\overline{x}^n}$$

Substituting this in the former equation we find for the value of ψx

$$\psi x = \frac{b}{1-a^2}\left\{ x^{p-n} + ax^{-p} \right\}$$

Ex. 3. Given the equation $x^n \psi x - x^m \psi \left(\frac{a-x}{1-cx}\right) = x^p$

by employing the same method its solution will be

$$\psi (x) = \frac{x^m \left(\frac{a-x}{1-cx}\right)^p + x^p \left(\frac{a-x}{1-cx}\right)^n}{\left(x \frac{a-x}{1-cx}\right)^n - \left(x \frac{a-x}{1-cx}\right)^m}$$

Problem XXXII.

Given the equation

$$F\left\{x, \psi x, \psi \alpha x, \ldots \psi \alpha^n x\right\} = 0$$

and also $\alpha^{n+1} = x$

putting successively $x, \alpha x, \alpha^2 x$, &c. $\alpha^n x$ for x, we have the following equations:

$$F\left\{x, \psi x, \psi \alpha x, \ldots \psi \alpha^n x\right\} = 0 \qquad (1)$$

$$F\left\{\alpha x, \psi \alpha x, \psi \alpha^2 x, \ldots \psi \alpha^n x, \psi x\right\} = 0 \qquad (2)$$

&c. &c.

$$F\left\{\alpha^n x, \psi \alpha^n x, \psi x, \psi \alpha x, \ldots \psi \alpha^{n-1} x\right\} = 0 \qquad (n+1)$$

From these $n+1$ equations we may eliminate the n quantities $\psi\alpha x$, $\psi\alpha^2 x$, and $\psi\alpha^n x$, and there will remain an equation of the form

$$F\left\{x,\ \alpha x,\ \alpha^2 x,\ \ldots\ \alpha^n x,\ \psi x\right\} = o$$

from which ψx may readily be found.

Ex. 1. Let $\psi x + \psi\alpha x = fx$ and $\alpha^3 x = x$

then $\psi\alpha x + \psi\alpha^2 x = f\alpha x$

also $\psi\alpha^2 x + \psi x = f\alpha^2 x$

hence we find $\psi x = \frac{1}{2}(fx - f\alpha x + f\alpha^2 x)$

Ex. 2. take the equation

$$\psi x + fx\,\psi\alpha x = f_1 x$$

where f and f_1 are perfectly arbitrary and $\alpha^2 x = x$; then making use of process above described we find

$$\psi x = \frac{f_1 x - fx\,f_1\alpha x}{1 - f\alpha x\,f\alpha x}$$

if $\alpha^3 x = x$ we should have

$$\psi x = \frac{f_1 x - fx\,f_1\alpha x + fx\,f\alpha x\,f_1\alpha^2 x}{1 - fx\,f\alpha x\,f\alpha^2 x}$$

and generally when $\alpha^n x = x$ we shall have

$$\psi x = \frac{f_1 x - fx\,f_1\alpha x + fx\,f\alpha x\,f_1\alpha^2 x - \&c. \pm fx\,f\alpha x\,..\,f\alpha^{n-2}x\,f_1\alpha^{n-1}x}{1 - fx\,f\alpha x\,f\alpha^2 x\,\ldots\,f\alpha^{n-1}x}$$

Ex. 3. Take the equation

$$\psi x\,\psi\alpha x + fx\,\psi x = f_1 x$$

if $\alpha^2 x = x$, ψx must be deduced from the equation

$$\psi x = \cfrac{f_1 x}{fx + \cfrac{f_1\alpha x}{f\alpha x + \psi x}}$$

and generally when $\alpha^n x = x$ the form of ψx is determined by the equation.

$$\psi x = \cfrac{\underset{1}{f} x}{fx + \cfrac{\underset{1}{f} \alpha x}{f\alpha x + \cfrac{\underset{1}{f} \alpha^2 x}{f\alpha^2 x + \&c. + \cfrac{\underset{1}{f} \alpha^{n-1} x}{f\alpha^{n-1} x + \psi x}}}}$$

It may be observed, that this method of discovering particular solutions by elimination, will not apply when the given equation contains only the different forms of the function without the variable quantity itself: thus it is not applicable to the equation

$$F\left\{\psi x,\ \psi\alpha x,\ ..\ \psi\alpha^n x\right\} = 0$$

the reason of this is obvious; for if we eliminate from this equation (by means of the n equations which arise by changing the order in which the functions are placed), all the functions but ψx, we shall have a result containing nothing but ψx and constant quantities, and therefore, ψx is equal to a constant quantity: it is true such a value of ψx will satisfy the equation, but it scarcely deserves the name of a solution.

Another exception is, when the equation

$$F\left\{x,\ \psi x,\ \psi\alpha x,\ \&c.\ \psi\,\alpha^n x\right\} = 0$$

is homogeneous relative to the different forms of the unknown function; for in this case when we attempt to eliminate them, they all disappear together, leaving an equation of condition; thus given

$$\psi x = (a - x)\,\psi\,\alpha\,x \text{ and } \alpha^2\,x = x$$

we have $\quad \psi\alpha x = (a - \alpha x)\,\psi\,\alpha^2 x = (a - \alpha x)\,\psi x$

and $\quad \psi x = (a - x)\,(a - \alpha x)\,\psi x$ or $1 = (a - \alpha x)\,(a - \alpha^2 x)$

which equation is not necessarily true.

Another exception is, when the given equation can be made to assume the form

$$F\left\{\overline{\psi x}, \overline{\psi \alpha x} \ldots \overline{\psi \alpha^n x}\right\} = fx$$

In this case the equation cannot be fulfilled unless fx is a symmetrical function of x, αx, &c. $\alpha^n x$, because the first side is such a symmetrical function: this reason, however, should be received with caution, for if the operation denoted by ψ be an inverse one, it may admit of several values, and it seems *possible*, that in such a case the condition relative to the form of f need not be fulfilled. In my former paper I explained the means of finding solutions of the equation $\psi^n x = x$. I then contented myself with explaining the theory without mentioning particular cases; as these latter may be required in our present enquiry, I shall subjoin the following particular solutions of $\psi^2 x = x$

$$\psi x = a - x \qquad \psi x = \log(a - \varepsilon^x) \qquad \psi x = (a^n - x^n)^{\frac{1}{n}} \qquad \psi x = \sqrt{1 - x^2}$$

$$\psi x = \frac{x-2}{x-1} \qquad \psi x = x - \log(\varepsilon^x - 1) \qquad \psi x = \frac{x}{(ax^n - 1)^{\frac{1}{n}}} \qquad \psi x = \frac{a}{x}$$

$$\psi x = \frac{1-x}{1+x} \qquad \psi x = \tan^{-1}(a - \tan x) \qquad \psi x = \frac{x}{x-1}$$

$$\psi x = \frac{a-bx}{b+cx} \qquad \psi x = \tan^{-1}\left(\frac{\sin(a-x)}{\cos a \cos x}\right) \qquad \psi x = \frac{x}{\sqrt{x^2-1}}$$

Particular cases of $\psi^3 x = x$

$$\psi x = \frac{a^2}{a-x} \qquad \psi x = \frac{\sqrt{ax^2 - a^2}}{x} \qquad \psi x = \frac{a+bx}{c - \frac{b^2+bc+c^2}{a}x}$$

$$\psi x = \frac{a^2}{ac - c^2 x} \qquad \psi x = \frac{(ax^n - a^2)^{\frac{1}{n}}}{x} \qquad \psi x = \log(a\varepsilon^x - a^2) - x$$

$$\psi x = \frac{ax - a^2}{x} \qquad \psi x = \left(\frac{a^2}{a - x^n}\right)^{\frac{1}{n}} \qquad \psi x = \log(\varepsilon^x - \varepsilon^c) - x + c$$

$$\psi x = \frac{1+x}{1-3x} \qquad \psi x = \frac{1}{1-x} \qquad \psi x = -\log(1 - \varepsilon^x)$$

Particular cases of $\psi^4 x = x$

$$\psi x = \frac{1}{2}\,\frac{1}{1-x} \qquad \psi x = \frac{1+x}{1-x} \qquad \psi x = \left(\frac{2}{2-x^n}\right)^{\frac{1}{n}}$$

$$\psi x = \frac{2}{2-x} \qquad \psi x = \frac{2a^2}{2ac-c^2x} \qquad \psi x = \frac{(2x^n-2)^{\frac{1}{n}}}{x}$$

$$\psi x = 2\,\frac{x-1}{x} \qquad \psi x = \frac{a+bx}{c-\frac{b^2+c^2}{2a}x} \qquad \psi x = \log 2 - x + \log(\varepsilon^x - 1)$$

Particular cases of $\psi^6 x = x$

$$\psi x = \frac{1}{3(1-x)} \qquad \psi x = \frac{3x-1}{3x} \qquad \psi x = \frac{a+bx}{c-\frac{b^2-bc+c^2}{3a}x}$$

$$\psi x = \frac{3}{3-x} \qquad \psi x = \frac{3a^2}{3ac-c^2x} \qquad \psi x = \frac{1}{x}\left(x^n - \frac{1}{3}\right)^{\frac{1}{n}}$$

$$\psi x = 3\,\frac{x-1}{x} \qquad \psi x = \frac{3+3x}{3-x} \qquad \psi x = \log 3 - x + \log(\varepsilon^x - 1)$$

Problem XXXIII.

Given the equation

$$\psi \alpha x = \frac{d\psi x}{dx}$$

α being such a function that $\alpha^2 x = x$

For x put αx, then

$$\psi\alpha^2 x = \psi x = \frac{d\psi\alpha x}{d\alpha x}$$

by differentiating this we have

$$\frac{d\psi x}{dx} = \frac{d.}{dx}\cdot\frac{d\psi\alpha x}{d\alpha x} \qquad (a)$$

but the left side of this equation is by the Problem equal to $\psi\alpha x$; therefore

$$\psi\alpha x = \frac{d.}{dx}\cdot\frac{d\psi\alpha x}{d\alpha x}$$

but we also have

$$\frac{d\psi\alpha x}{dx} = \frac{d\psi\alpha x}{d\alpha x}\cdot\frac{d\alpha x}{dx}$$

consequently

$$\frac{d\psi\alpha x}{d\alpha x} = \frac{d\psi\alpha x}{dx}\cdot\left(\frac{d\alpha x}{dx}\right)^{-1}$$

this being substituted in (a) gives

$$\psi\alpha x = \frac{d}{dx} \cdot \left\{ \frac{d\,\psi\,\alpha\,x}{dx} \left(\frac{d\,\psi\,\alpha\,x}{dx}\right)^{-1} \right\}$$

put $\psi\alpha x = z$ then

$$z = \frac{d.}{dx} \cdot \left(\frac{dz}{dx} \left(\frac{d\,\alpha\,x}{dx}\right)^{-1} \right)$$

which is a differential equation, from whose solution z or $\psi\alpha x$ may be found.

Ex. 1. Given the equation

$$\psi\,(a - x) = \frac{d\,\psi\,x}{dx}$$

in this case $\frac{d\,\alpha\,x}{dx} = -1$, and the differential equation is

$$z\,dx^2 + d^2 z = 0$$

its integral is $z = \psi\,(a - x) = b \cos x + c \sin x$.

The two constant quantities which have entered by integration must be determined so as to satisfy the original equation. This condition gives

$$c = \frac{-b \cos a}{1 - \sin a}$$

the quantity b still remaining arbitrary; the solution of the equation $\psi\,(a - x) = \frac{d\,\psi\,x}{dx}$ is therefore

$$\psi\,x = b \cos\,(a - x) - \frac{b \cos a}{1 - \sin a} \sin\,(a - x)$$

Ex. 2. Take the equation

$$\psi \frac{1}{x} = \frac{d\,\psi\,x}{dx}$$

in this case $\alpha x = \frac{1}{x}$ and $\frac{d\,\alpha\,x}{dx} = \frac{-1}{x^2}$

and the differential equation becomes

$$z\,dx^2 + 2\,x\,dx\,dz + x^2\,d^2 z = 0$$

whose solution is

$$z = \psi \frac{1}{x} = -\frac{b}{2p+1} x^{-p-1} + \underset{1}{b}\,x^{p} \qquad p = \frac{-1 \pm \sqrt{-3}}{2}$$

and $\psi x = -\frac{b}{2p+1} x^{p+1} + \underset{1}{b}\,x^{-p}$

in order to determine the constants b and b_{1}, substitute this expression in the given equation and it will be found that

$$-\frac{b}{2p+1} = -b_{1}p \text{ therefore}$$

$$\psi x = -b_{1}px^{p+1} + b_{1}x^{-p}$$

in which there still remains one arbitrary constant.

It is observable that both these solutions contain one constant. Let us suppose this to be changed into an arbitrary function of x, and let us determine what conditions it must be subject to, that it may satisfy the Problem: taking the second example we have

$$\psi x = (x^{-p} - px^{p+1})\,\phi x$$

and the equation becomes

$$(x^{p} - px^{-p-1})\,\phi\frac{1}{x} = (x^{-p} - px^{p+1})\frac{d\phi x}{dx} + \frac{d(x^{-p} - px^{p+1})}{dx}\phi x$$

from this equation ϕx must be determined (the method of doing which will appear in a subsequent Problem). If this solution contains an arbitrary constant, the same process may be again repeated. We may thus continue deducing one solution from another as long as we can solve the differential equations to which they give rise, but still these will only be particular solutions.

Problem XXXIV.

Given the equation

$$\psi\alpha x = \frac{d^{n}\psi x}{dx^{n}}$$

and $\alpha^{p}x = x$, put for x successively αx, $\alpha^{2}x$, . . $\alpha^{p-1}x$ then we have

$$\psi\alpha x = \frac{d^{n}\psi x}{dx^{n}}$$

$$\psi\alpha^2 x = \frac{d^n\,\psi\alpha x}{(d\alpha x)^n}$$

&c. &c.

$$\psi\alpha^{p-1} x = \frac{d^n\psi\alpha^{p-2}x}{(d\alpha^{p-2}x)^n}$$

$$\psi\alpha^p x = \frac{d^n\psi\alpha^{p-1}x}{(d\alpha^{p-1}x)^n}$$

but $\psi\alpha^p x = \psi x$ and combining all these equations we have

$$\psi\alpha^p x = \psi x = \frac{d.^n}{(d\alpha^{p-1}x)^n} \cdot \frac{d.^n}{(d\alpha^{p-2}x)^n} \cdots \frac{d^n\,\psi x}{(dx)^n}$$

which is a differential equation of the pn^{th} order and putting $\psi x = z$ we have

$$z = \frac{d^{np}z}{(d\alpha^{p-1}x.\,d\alpha^{p-2}x \ldots d\alpha x.\,dx)^n}$$

this being integrated gives the value of z or ψx.

PROBLEM XXXV.

Given the equation

$$F\left\{x,\ \psi x,\ \psi\alpha x,\ \frac{d\psi x}{dx}\right\} = 0 \quad \text{also } \alpha^p x = x$$

Find the value of $\psi\alpha x$ from this equation and substitute in it αx, $\alpha^2 x$, &c. $\alpha^{p-1} x$ for x, then we have

$$\psi\alpha x = F_1\left\{x,\ \psi x,\ \frac{d\psi x}{dx}\right\}$$

$$\psi\alpha^2 x = F_1\left\{\alpha x,\ \psi\alpha x,\ \frac{d\psi\alpha x}{d\alpha x}\right\}$$

&c. &c.

$$\psi\alpha^p x = F_1\left\{\alpha^{p-1}x,\ \psi\alpha^{p-1}x,\ \frac{d\alpha^{p-1}x}{d\alpha^{p-1}x}\right\}$$

In each of these equations for $\frac{d\psi\alpha x}{d\alpha x}$, $\frac{d\psi\alpha^2 x}{d\alpha^2 x}$, &c. $\frac{d\psi\alpha^{p-1}x}{d\alpha^{p-1}x}$

put their values $\frac{d\psi\alpha x}{dx}\left(\frac{d\alpha x}{dx}\right)^{-1}$, $\frac{d\psi\alpha^2 x}{dx}\left(\frac{d\alpha^2 x}{dx}\right)^{-1}$, &c. $\frac{d\psi\alpha^{p-1}x}{dx}\left(\frac{d\alpha^{p-1}x}{dx}\right)^{-1}$

and also differentiate the results. Then we shall have the two following sets of equations

$$\psi\alpha\, x = F_{1}\left\{x,\ \psi x,\ \frac{d\psi x}{dx}\right\} \qquad (1)$$

$$\psi\alpha^2 x = F_{1}\left\{\alpha x,\ \psi\alpha x,\ \frac{d\psi\alpha x}{dx}\left(\frac{d\alpha x}{dx}\right)^{-1}\right\} \qquad (2)$$

&c. &c.

$$\psi\alpha^{p-1} x = F_{1}\left\{\alpha^{p-2}x,\ \psi\alpha^{p-2}x,\ \frac{d\psi\alpha^{p-2}x}{dx}\left(\frac{d\alpha^{p-2}x}{dx}\right)^{-1}\right\} \qquad (p-1)$$

$$\psi\alpha^{p} x = F_{1}\left\{\alpha^{p-1}x,\ \psi\alpha^{p-} x,\ \frac{d\psi\alpha^{p-1}x}{dx}\left(\frac{d\alpha^{p-1}a}{dx}\right)^{-1}\right\} \qquad (p)$$

and also

$$\frac{d\psi\alpha x}{dx} = \frac{d.}{dx} F_{1}\left\{x,\ \psi x,\ \frac{d\psi x}{dx}\right\} \qquad (\alpha, 1)$$

$$\frac{d\psi\alpha^2 x}{dx} = \frac{d.}{dx} F_{1}\left\{\alpha x,\ \psi\alpha x,\ \frac{d\psi\alpha x}{dx}\left(\frac{d\alpha x}{dx}\right)^{-1}\right\} \qquad (\alpha, 2)$$

&c. &c.

$$\frac{d\psi\alpha^{p-1}x}{dx} = \frac{d.}{dx} F_{1}\left\{\alpha^{p-1}\psi\alpha^{p-2}x,\ \frac{d\psi\alpha^{p-2}x}{dx}\left(\frac{d\alpha^{p-2}x}{dx}\right)^{-1}\right\} \qquad (\alpha, p-1)$$

$$\frac{d\psi\alpha^{p}x}{dx} = \frac{d.}{dx} F_{1}\left\{\alpha^{p-1}x,\ \psi\alpha^{p-1}x,\ \frac{d\psi\alpha^{p-1}x}{dx}\left(\frac{d\alpha^{p-1}x}{dx}\right)^{-1}\right\} \qquad (\alpha, p)$$

Since $\alpha^p x = x$ equation (p) becomes

$$\psi x = F_{1}\left\{\alpha^{p-1}x,\ \psi\alpha^{p-1}x,\ \frac{d\psi\alpha^{p-1}x}{dx}\left(\frac{d\alpha^{p-1}x}{dx}\right)^{-1}\right\}$$

from this by means of equations $(p-1)$ and $(\alpha, p-1)$ we may eliminate $\psi\alpha^{p-1}x$, and $\frac{d\psi\alpha^{p-1}x}{dx}$ the resulting equation will contain only $x, \psi x, \psi\alpha x$, &c. $\psi\alpha^{p-2}x$ and their differentials. From this by means of $(p-2)$ and $(\alpha, p-2)$ we may eliminate $\psi\alpha^{p-2}x$, and its differential, leaving an equation containing only $x, \psi x, \psi\alpha x$, &c. $\psi\alpha^{p-3}x$ and their differentials. In the same manner $\psi\alpha^{p-3}x$ may be eliminated, and the process may be continued

until the last equation will only contain x, ψx and their differentials; this equation must be integrated, and it will determine the value of ψx in terms of x.

The same method may be employed for the solution of the much more general equation

$$F\left\{x, \psi x, \psi\alpha x, \ldots \psi\alpha^{p-1}x, \frac{d^n \psi\alpha x}{dx^n}, \frac{d^m\psi\alpha^2 x}{dx^m}, \&c.\right\} = o$$

provided also, that $\alpha^p x = x$.

By substituting successively for x the quantities αx, $\alpha^2 x$, &c. $\alpha^{p-1}x$, we shall have p equations containing the functions ψx, $\psi\alpha x$, and $\psi\alpha^{p-1}x$ and their differentials.

Let each of these be differentiated as often as may be required, and we shall have two sets of equations by means of which all the quantities except x and ψx, and their differentials may be eliminated, the result is a common differential equation whose integral will afford the value of ψx in terms of x. If after satisfying the conditions of the Problem, there remain any arbitrary constants, we may suppose them functions of x, and new equations will thence arise by which they may be determined.

It might occur (when there are several arbitrary quantities) that, by assigning particular values to some of them, the others might remain in a certain degree arbitrary, should this be the case, we should obtain general solutions.

Problem XXXVI.

Given the equation

$$\psi^2 x = \frac{d\psi x}{dx}$$

Assume $\psi x = \overset{-1}{\phi} f \phi x$, then the equation becomes

$$\overset{-1}{\phi} f^2 \phi x = \frac{d\overset{-1}{\phi} f\phi x}{dx}$$

putting $\overset{-1}{\varphi}$ for x we have

$$\overset{-1}{\varphi} f^2 x = \frac{d\overset{-1}{\varphi} fx}{d\overset{-1}{\varphi} x} = \frac{d\overset{-1}{\varphi} fx}{dx} \left(\frac{d\overset{-1}{\varphi} x}{dx}\right)^{-1}$$

or

$$\overset{-1}{\varphi} f^2 x . \frac{d\overset{-1}{\varphi} x}{dx} = \frac{d\overset{-1}{\varphi} fx}{dx}$$

Some particular solution of the original equation must now be assumed as the value of f, and the resulting differential functional equation must be solved. The only particular case of the equation $\psi^2 x = \frac{d\psi x}{dx}$ with which I am at present acquainted, is

$$\psi x = \left(\frac{1 \pm \sqrt{-3}}{2}\right)^{\frac{1 \mp \sqrt{-3}}{2}} \times x^{\frac{1 \pm \sqrt{-3}}{2}}$$

Other more complicated equations containing the various orders of functions, and their differentials may be reduced to those of the first order by the same means, but great difficulties still remain; it is by no means easy to discover particular solutions of the original equations, and even when these are found, the functional equations of the first order which remain to be solved, are of considerable difficulty. I shall therefore refrain from giving any more examples, and proceed to show how functional equations involving definite integrals may be reduced to those we have already treated. Such equations might occur in a variety of curious and interesting enquiries, few of which have yet been noticed. D'ALEMBERT, in one of the volumes of his Opuscules, has examined a question which may be referred to this class; it is the following. Suppose a sphere composed of particles of matter, what must be the law of attraction amongst these particles, so that the force of the whole sphere acting on a particle at a distance, may follow the same law? the question might be varied by supposing the law to be given, and the form of the solid to be required;

but the general solution of such questions is by no means easy.

PROBLEM XXXVII.

Required the nature of the function ψ such that

$$\int dx\ \psi^2 x = \psi a$$

the integral being taken between the limits $x = o$ and $x = a$.

Assume $\varphi\,(x, v)$ such that

$$\varphi\,(x, v) - \varphi\,(o, v) = v$$

the form of φ may be ascertained from this equation by means already described. Then if we make

$$\int dx\ \psi^2 x = \varphi\,(x, \psi a) \qquad (1)$$

it is evident that between the two limits $x = o$ and $x = a$, the integral will be reduced to ψa, and we have therefore a differential functional equation whose mode of solution has already been pointed out. Other more complicated equations may be solved in the same way; these I shall omit. I shall, however, make some observations on this method of solution, with a view to point out some questions of considerable importance.

In equation (1) the function indicated by φ is so assumed that we may have

$$\varphi\,(a, \psi a) - \varphi\,(o, \psi a) = \psi a$$

from which, perhaps, it might be imagined, that $\varphi\,(x, \psi a)$ must contain only x, ψa and constant quantities, but the condition would still be fulfilled if it contained $\psi^2 a$, $\psi^3 a$, or $\psi^n a$, which though not actually variable cannot strictly be regarded as constant. To fix our ideas, let us consider the example in this Problem; one value of $\varphi\,(x, \psi a)$ is evidently $\varphi\,(x, \psi a) = \frac{x^n}{a^n}\,\psi a$, we have therefore

$$\int dx\ \psi^2 x = \frac{x}{a^n}\,\psi a$$

and by differentiating

$$\psi^2 x = \frac{nx^{n-1}}{a^n}\,\psi a$$

from which ψx may be found.

This is a solution derived from a certain form attributed to ϕ, but we might also give to ϕ the form

$$\phi\,(x,\,\psi a) = \frac{x^n}{a^n}\psi a + x^p\,(x-a)^q f\,(a,\,\psi a,\,\psi^2 a,\,..\,\psi^n a)$$

and, in that case, the equation to be solved would be

$$\psi^2 x = \frac{nx^{n-1}}{a^n}\,\psi a + \frac{d.}{dx}\,(x\,(x-a)^q)\,f\,(a,\,\psi a,\,\psi^2 a,\,..\,\psi^n a\,))$$

this contains only the second function of the unknown quantity and must be solved as a second functional equation, considering $a, \psi a$, &c. $\psi^n a$ as constant quantities; let its solution be

$$\psi x = \mathrm{F}\left\{x,\,a,\,\psi a,\,..\,\psi^n a\right\} \qquad (a)$$

then we must put $x=a$ and determine ψa from the equation

$$\psi a = \mathrm{F}\left\{a,\,a,\,\psi a,\,..\,\psi^n a\right\}$$

the value of ψa thus deduced, will furnish the values of $\psi^2 a$, &c. $\psi^n a$, and these being substituted in a, will give the value of ψx; this solution is evidently of a different nature from the former, and forms another species.

Again, the following form of ϕ will also agree with the conditions

$$\phi\,(x,\psi a) = \frac{x^n}{a^n}\psi a + x^p(x-a)^q f\left\{a,\psi a,\psi^2 a,..\,\psi^n a,\,x,\psi x,\psi^2 x,...\psi^k x\right\}$$

which being substituted in the Problem ψx must be found from a functional equation of the k^{th} order; x must then be put equal to a, and the new functional equation of the n^{th} order relative to a must be solved; this is a third species of solution different from either of the former. Respecting these three species of solutions, a very important question

may be proposed. What degree of generality does each possess, and how many and what sort of arbitrary functions does each solution involve? To discuss this question, and to point out the nature of other solutions yet more general, which may be found for these and other similar Problems, would far exceed the limits of a mere outline of the calculus. I shall conclude my remarks on this Problem by stating the plan to be pursued in one particular case, which may serve as a model for all similar operations. Take as the form of $\phi\,(x, \psi a)$

$$\phi\,(x, \psi a) = x\,(x-a)\frac{\psi x\,\psi^2 x}{\psi^2 o\,\psi^2 a} - \frac{a-x}{a}\frac{\psi x}{\psi o}\psi a$$

then we have

$$\psi^2 x = \frac{d.}{dx}\left\{x\,(x-a)\frac{\psi x\,\psi^2 x}{\psi^2 o\,\psi^2 a} - \frac{a-x}{a}\frac{\psi x}{\psi o}\psi a\right\}$$

this is a differential functional equation which must be solved on the hypothesis of a, ψa, $\psi^2 a$, ψo and $\psi^2 o$, being constant quantities. Let its solution be

$$\psi x = F\left\{x, a, \psi a, \psi^2 a, \psi o, \psi^2 o\right\} \qquad (1)$$

we must now put $x=a$ and treat the resulting equation as one of the second order, considering ψo and $\psi^2 o$ as constants. Let its solution be

$$\psi a = \underset{1}{F}\left\{a, \psi o, \psi^2 o\right\} \qquad (2)$$

Now substitute o for a retaining o as a letter instead of making it actually zero, there will result a new functional equation of the second order, whose solution is

$$\psi o = \underset{2}{F}\left\{\,o\,\right\}$$

and lastly, substituting this value of ψo, and also that of $\psi^2 o$ which may be deduced from it in (2) we have the value of ψa, from this $\psi^2 a$ may be found, and these being substituted in (1) give the value of ψx.

Problem XXXVIII.

Given the equation

$$\psi(x,y) = \frac{d\psi(x,\alpha y)}{dx}$$

where $\alpha^2 y = y$.

For y put αy and the equation becomes

$$\psi(x,\alpha y) = \frac{d\psi(x,\alpha^2 y)}{dx} = \frac{d\psi(x,y)}{dx}$$

differentiate this relative to x, then we have

$$\frac{d\psi(x,\alpha y)}{dx} = \frac{d^2\psi(x,y)}{dx^2}$$

this substituted in the original equation, gives

$$\psi(x,y) = \frac{d^2\psi(x,y)}{dx^2}$$

which is a partial differential equation, whose solution is

$$\psi(x,y) = \varepsilon^x \phi y + \varepsilon^{-x} \underset{1}{\phi} y$$

ϕy and $\underset{1}{\phi} y$ being two arbitrary functions of y, so constituted as to fulfil the original equation. These may thus be determined, since

$$\psi(x,y) = \varepsilon^x \phi y + \varepsilon^{-x} \underset{1}{\phi} y$$

we have $$\frac{d\psi(x,\alpha y)}{dx} = \varepsilon^x \phi\alpha y - \varepsilon^{-x} \underset{1}{\phi}\alpha y$$

and, since these two quantities must be equal, we have the following equations

$$\phi y = \phi\alpha y \text{ and } \underset{1}{\phi} y = -\underset{1}{\phi}\alpha y$$

the former of these is easily satisfied by putting for ϕy any symmetrical function of y and αy; and a particular solution of the latter is

$$\underset{1}{\phi} y = (-y + \alpha y)\, c$$

and since this solution contains an arbitrary constant, it may

be changed by Prob. VIII. Part I. into any arbitrary function which does not vary when y becomes αy; its general solution is therefore

$$\varphi_2 y = (-y + \alpha y)\, \varphi\, (\overline{y}, \overline{\alpha y})$$

and consequently the general solution of the equation of this Problem is

$$\psi\,(x, y) = \varepsilon^{x}\, \varphi\, (\overline{y}, \overline{\alpha y}) + \varepsilon^{-x}\, (\alpha y - y)\, \varphi_1\, (\overline{y}, \overline{\alpha y})$$

Ex. 1. Take the equation

$$\psi\,(x, y) = \frac{d\psi\,(x, a-y)}{dx}$$

in this case $\alpha y = a - y$, and the general solution is

$$\psi\,(x, y) = \varepsilon^{x}\, \varphi\, (\overline{y}, \overline{a-y}) + \varepsilon^{-x}\, (a - 2y)\, \varphi_1\, (\overline{y}, \overline{a-y})$$

Ex. 2. Let the given equation be

$$\psi\,(x, y) = \frac{d\psi\,(x, \frac{1}{y})}{dx}$$

here $\alpha y = \frac{1}{y}$ and the general solution is

$$\psi\,(x, y) = \varepsilon^{x}\, \varphi\, (\overline{y}, \overline{\tfrac{1}{y}}) + \varepsilon^{-x}\, \frac{1-y^2}{y}\, \varphi_1\, (\overline{y}, \overline{\tfrac{1}{y}})$$

PROBLEM XXXIX.

Given the equation

$$\psi\,(x, y) = \frac{d\psi\,(x, \alpha y)}{dx}$$

supposing $\alpha^{p}\, y = y$.

By substituting successively αy, $\alpha^2 y$, &c. $\alpha^{p-1} y$ for y, we have the following equations

$$\psi\,(x,\ y) = \frac{d\psi\,(x, \alpha y)}{dx}$$

$$\psi\,(x, \alpha y) = \frac{d\psi\,(x, \alpha^2 y)}{dx}$$

&c. &c.

$$\psi\,(x, \alpha^{p-1} y) = \frac{d\psi\,(x, \alpha^{p} y)}{dx}$$

From the first of these we may eliminate $\frac{d\psi(x,\,\alpha y)}{dx}$ by means of the differential of the second, and from the result $\frac{d\psi(x,\,\alpha^2 y)}{dx}$ may be eliminated by means of the differential of the third. And by continuing this process, observing that $\psi\,(x, \alpha^p y) = \psi\,(x, y)$ we shall find

$$\psi\,(x, y) = \frac{d^p\psi\,(x, y)}{dx^p}$$

this partial differential equation must be solved, and the arbitrary functions which enter into its integral, must be made to satisfy the conditions of the Problem.

Ex. Let $p=4$, then $\psi\,(x, y) = \frac{d\psi\,(x,\,\alpha y)}{dx}$ and the solution of the resulting partial differential equation will be

$$\psi\,(x, y) = \varepsilon^x \underset{11}{\varphi} y - \varepsilon^{-x} \underset{12}{\varphi} y + \sin x\,.\,\underset{13}{\varphi} y + \cos x\,.\,\underset{14}{\varphi} y$$

hence

$$\frac{d\psi(x,\,\alpha y)}{dx} = \varepsilon^x \underset{11}{\varphi} \alpha y + \varepsilon^{-x} \underset{12}{\varphi} \alpha y + \cos x\,.\,\underset{13}{\varphi} \alpha y - \sin x\,.\,\underset{14}{\varphi} \alpha y$$

the first condition to be satisfied is

$$\underset{11}{\varphi} y = \underset{11}{\varphi} \alpha y$$

which is readily fulfilled by making $\underset{11}{\varphi} y = \underset{1}{\varphi}\,(\overline{y}, \overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y})$, the next condition is

$$\underset{12}{\varphi} y = -\,\underset{12}{\varphi} \alpha y$$

This must be solved by Prob. VIII. Part I., and we shall have

$$\underset{12}{\varphi} y = (-y + \alpha y - \alpha^2 y + \alpha^3 y)\,\underset{2}{\varphi}\,(\overline{y}, \overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y})$$

the third and fourth conditions are

$$\underset{13}{\varphi} y = -\,\underset{14}{\varphi} \alpha y \text{ and } \underset{14}{\varphi} y = \underset{13}{\varphi} \alpha y$$

In the second of these put αy for y, and it becomes $\underset{14}{\varphi} \alpha y = \underset{13}{\varphi} \alpha^2 y$,

this substituted in the former, gives

$$\underset{13}{\varphi} y = -\underset{13}{\varphi}\alpha^2 y$$

whose general solution being found by the method in the first part gives

$$\underset{13}{\varphi} y = (-y + \alpha^2 y)\ \underset{13}{\varphi}(\overline{y}, \overset{1}{\overline{\alpha y}}, \overline{\alpha^2 y}, \overset{1}{\overline{\alpha^3 y}})$$

and consequently

$$\underset{14}{\varphi} y = (-\alpha y + \alpha^3 y)\ \underset{3}{\varphi}(\overline{\alpha y}, \overset{1}{\overline{\alpha^2 y}}, \overline{\alpha^3 y}, \overset{1}{\overline{y}})$$

these values being respectively substituted, we have for the general solution of the Problem in this example,

$$\psi(x,y) = \varepsilon^{x}\varphi(\overline{y}, \overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y}) + \varepsilon^{-x}(-y + \alpha y - \alpha^2 y + \alpha^3 y)\underset{2}{\varphi}(\overline{y}, \overline{\alpha y}, \overline{\alpha^2 y}, \overline{\alpha^3 y}) +$$
$$+ (-y + \alpha^2 y)\underset{2}{\varphi}(\overline{y}, \overset{1}{\overline{\alpha y}}, \overline{\alpha^2 y}, \overset{1}{\overline{\alpha^3 y}}) \sin x + (-\alpha y + \alpha^3 y)\underset{3}{\varphi}(\overline{\alpha y}, \overset{1}{\overline{\alpha^2 y}}, \overline{\alpha^3 y}, \overset{1}{\overline{y}}) \cos x$$

If the original equation had been

$$\psi(x, y) = \frac{d^n \psi(x, \alpha y)}{dx^n}$$

the partial differential equation to be solved would have been $\psi(x, y) = \frac{d^{np}\psi(x, y)}{dx^{np}}$.

This form is rather remarkable, the equation can always be integrated when np is a whole number; let us suppose n to be a fraction and p a whole number, some multiple of the denominator of n.

Ex. Let $n = \frac{1}{2}, p = 2$, then $np = 1$, and $\alpha^2 y = y$, and the equation to be solved is

$$\psi(x, y) = \frac{d^{\frac{1}{2}}\psi(x, \alpha y)}{dx^{\frac{1}{2}}}$$

whose solution is $\psi(x, y) = \varepsilon^{x}\ \underset{1}{\varphi} y$, or by assigning a proper form to $\underset{1}{\varphi} y$ it becomes

$$\psi(x, y) = \varepsilon^{x}\varphi(\overline{y}, \overline{\alpha y})$$

Not only may the index of differentiation become fractional,

but the index of the order of a function may be a fraction* or even a variable quantity, and such equations as the following might occur

$$\frac{d^{\frac{1}{2}}\psi^{\frac{1}{2}}x}{\sqrt{dx}} = \frac{d^{\frac{1}{3}}\psi^{n}x}{dn^{\frac{1}{3}}}$$

To notice the extreme difficulty of the enquiries to which such equations would lead, might seem superfluous, though it may not be deemed equally so to support my own opinion of their utility by the authority of one well acquainted with these subjects. LACROIX, in the third volume of his Traité du Calcul, Diff. et Int. speaking of fractional indices of differentiation, observes, "L'Analyse offre une foule d'expressions de ce genre, qui tiennent presque toutes aux théories les plus importantes et les plus délicates, et les réflexions que j'ai exposées dans le No. 965, me portent à croire que leur considération peut contribuer beaucoup aux progrès de la science du calcul."

PROBLEM XL.

Given the equation

$$\frac{d\,\psi(x, \beta y)}{dx} = \frac{d\,\psi(\alpha x, y)}{dy}$$

also $\alpha^2 x = x$ and $\beta^2 y = y$.

Put αx for x, and βy for y, then the equation becomes

$$\frac{d\psi(\alpha x, y)}{d\alpha x} = \frac{d\psi(x, \beta y)}{d\beta y}$$

$$\text{hence } \frac{d\psi(\alpha x, y)}{dx}\left(\frac{d\alpha x}{dx}\right)^{-1} = \frac{d\psi(x, \beta y)}{dy}\left(\frac{d\beta y}{dy}\right)^{-1}$$

differentiate this equation relative to y, and the original one relative to x: then the two results are

$$\frac{d^2\psi(\alpha x, y)}{dx dy} = \frac{d\alpha x}{dx}\,\frac{d.}{dy}\left\{\frac{d\psi(x, \beta y)}{dy}\left(\frac{d\beta y}{dy}\right)^{-1}\right\}$$

* The difficulties which occur in treating functions with negative indices are similar to those in which they are positive; it may however be observed, that from the notation we have established, the following consequences follow:

$$\psi^{0,1}(x, y) = x \text{ and } \psi^{1,0}(x, y) = y$$

and generally $\quad \psi^{0,n}(x, y) = x$ and $\psi^{n,0}(x, y) = y$

also $\psi^{0,0}(x, x) = x$ and if $\psi^{1,1}(x, y) = v$, then we have

$$x = \overset{-}{\psi}{}^{1,1}(v, y) \text{ and also } = y\,\psi^{1,-1}(x, v)$$

and $$\frac{d^2\psi(\alpha x, y)}{dx dy} = \frac{d^2\psi(x, \beta y)}{dx^2}$$

hence $$\frac{d^2\psi(x, \beta y)}{dx^2} = \frac{d\alpha x}{dx} \cdot \frac{d.}{dy}\left\{\frac{d\psi(x, \beta y)}{dy}\left(\frac{d\beta y}{dy}\right)^{-1}\right\}$$

put βy for y, observing that $\left(\frac{dy}{d\beta y}\right)^{-1} = \left\{\frac{dy}{dy}\left(\frac{d\beta y}{dy}\right)^{-1}\right\}^{-1} = \frac{d\beta y}{dy}$

and also $\frac{d\psi(x, y)}{d\beta y} = \frac{d\psi(x, y)}{dy}\left(\frac{d\beta y}{dy}\right)^{-1}$, then there will result the equation

$$\frac{d\beta y}{dy}\,\frac{d^2\psi(x, y)}{dx^2} = \frac{d\alpha x}{dx}\,\frac{d^2\psi(x, y)}{dy^2}$$

This is a partial differential equation from whose solution $\psi(x, y)$ may be found.

Ex. 1. Given the equation $\frac{d\psi(a-x, y)}{dy} = \frac{d\psi(x, b-y)}{dx}$ in this case $\alpha x = a - x$ and $\beta y = b - y$, and the differential equation to be solved is

$$\frac{d^2\psi(x, y)}{dx^2} = \frac{d^2\psi(x, y)}{dy^2}$$

and its solution is

$$\psi(x, y) = \varphi(x + y) + \varphi_1(x - y)$$

the two arbitrary functions φ and φ_1 must be determined so as to fulfil the given equation, for which purpose we have

$$\frac{d\psi(a-x, y)}{dy} = \varphi'(a - x + y) - \varphi_1'(a - x - y)$$

and $$\frac{d\psi(x, b-y)}{dx} = \varphi'(b + x - y) + \varphi_1'(-b + x + y)$$

φ' and φ_1' being respectively the differential coefficients of φ and φ_1, since these two expressions must be equal, we have

$$\varphi'(a - \overline{x - y}) = \varphi'(b + \overline{x - y})$$

and $$-\varphi_1'(a - \overline{x + y}) = \varphi_1'(-b + \overline{x + y})$$

whose solutions are

$$\overset{\prime}{\varphi}(x-y) = \chi\left\{\overline{x-y},\ \overline{a+b-x+y}\right\}$$

and $$\underset{1}{\overset{\prime}{\varphi}}(x+y) = (a-b-2x-2y)\underset{1}{\chi}\left\{\overline{x+y},\ \overline{a-b-x-y}\right\}$$

hence the general solution of the equation

$$\frac{d\psi(a-x,y)}{dy} = \frac{d\psi(x,b-y)}{dx}$$

is $$\psi(x,y) = \int(dx+dy)\,\chi\left\{\overline{x+y},\ \overline{a+b-x-y}\right\} +$$
$$+\int(dx-dy)\,(a-b-2x+2y)\,\underset{1}{\chi}\left\{\overline{x-y},\overline{a-b-x+y}\right\}$$

Ex. 2. Given the equation

$$\frac{d\psi\left(x,\frac{1}{y}\right)}{dx} = \frac{d\psi\left(\frac{1}{x},y\right)}{dy}$$

the partial differential equation to be solved is in this case

$$\frac{d\psi(x,y)}{dy^2} = \frac{x^2}{y^2}\,\frac{d\psi(x,y)}{dx^2}$$

and its solution is

$$\psi(x,y) = \quad\left(\frac{x}{y}\right) + \underset{1}{\varphi}(xy)$$

determining φ and $\underset{1}{\varphi}$ so as to fulfil the conditions of the equation, we have

$$\psi(x,y) = c(x+y) + \int d(xy)\,.\left(\frac{1}{xy}\right)^{\frac{1}{2}}\chi\left\{\overline{xy},\overline{\frac{1}{xy}}\right\}$$

Problem XLI.

Given the equation

$$F\left\{x, y, \psi(x,y), \psi(\alpha x, y), \&c.\ \frac{d^n\psi(x,\beta y)}{dx^n},\ \frac{d^k\psi(\alpha^2 x, y)}{dy^k},\ \&c.\right\} = o$$

and let $\alpha^p x = x$ and $\beta^q y = y$,

then there may be pq different forms of the function ψ contained in the general expression $\psi(\alpha^r x, \beta^s y)$, r varying from o to $p-1$, and s varying from o to $q-1$.

In the first place it may be observed, that if we substitute αx for x in such a quantity as

$$\frac{d^n\psi(\alpha^3 x, \beta y)}{dx^n}$$

we shall have

$$\frac{d^n \psi (\alpha^4 x, \beta y)}{(d\alpha x)^n}$$

which may always be reduced to the form

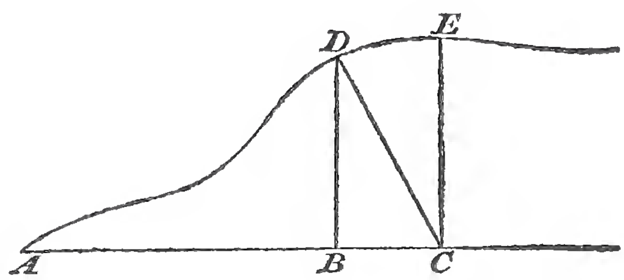
$$f(x) \frac{d^n \psi (\alpha^4 x, \beta y)}{dx^n}$$

If now in the original equation we substitute successively αx, $\alpha^2 x$, . . $\alpha^{p-1} x$ for x, and βy, $\beta^2 y$, &c. $\beta^{q-1} y$ for y, we shall have pq equations containing pq forms of the unknown function and their differentials. By means of these pq equations and the differentials of them, we may eliminate all the different forms of the function ψ, except one: let the one which remains be $\psi (x, y)$, then we have an equation of partial differentials containing only x, y, $\psi (x, y)$ and their differentials: and from the solution of this equation $\psi (x, y)$ may be found; a certain number of arbitrary functions will be contained in this integral; these must all be determined so as to satisfy the original equation.

Amongst the numerous questions to which the calculus of functions is applicable, I shall select a problem proposed by EULER in one of the volumes of the Acta Acad. Petrop. as it will offer an example of a mode of treating of functional equations of a nature yet more general than those contained in this paper.

Problem XLII.

Required the nature of a curve such that taking any ordinate



DB, and drawing a normal at the point D, the next ordinate CE raised at the foot of the normal shall be equal to that normal.

Let AB $= x$, BD $= y$ and $y = \psi x$ be the equation of the curve, then BC $= \frac{ydy}{dx}$,

and DC $= \sqrt{y^2 + \left(\frac{y\,dy}{dx}\right)^2}$ and by the condition of the Problem we have

$$\sqrt{y^2 + \left(\frac{ydy}{dx}\right)^2} = \psi\left(x + \frac{ydy}{dx}\right)$$

hence

$$[\psi x]^2 + \left[\frac{\psi x d\psi x}{dx}\right]^2 = \left[\psi\left(x + \frac{\psi x d\ x}{dx}\right)\right]^2 \qquad (a)$$

This is apparently a very difficult functional equation, and I am not acquainted with any direct method of solving other similar ones. It is in fact only from a peculiar condition which this equation involves that any solutions have been obtained, the condition to which I allude is, that the quantity $\frac{\psi x\, d\psi x}{dx}$ does not change, when for x we substitute $x + \frac{\psi x\, d\psi x}{dx}$ or expressed in symbols, that

$$\frac{\psi\left(x+\frac{\psi x\, d\psi x}{dx}\right)d\,.\,\psi\left(x+\frac{\psi x\, d\psi x}{dx}\right)}{d\left(x+\frac{\psi x\, d\psi x}{dx}\right)}=\frac{\psi x\, d\psi x}{dx}$$

which may be thus proved differentiate (a) which gives

$$\psi\left(x+\frac{\psi x\, d\psi x}{dx}\right)d\,.\,\psi\left(x+\frac{\psi x\, d\psi x}{dx}\right)=\psi x\, d\psi x+\frac{\psi x\, d\psi x}{dx}\,d\,.\,\frac{\psi x\, d\psi x}{dx}$$

and by dividing both sides of this equation by

$$d\left(x+\frac{\psi x\, d\psi x}{dx}\right)=dx+d\,.\left(\frac{\psi x\, d\psi x}{dx}\right)$$

we have

$$\frac{\psi\left(x+\frac{\psi x\, d\psi x}{dx}\right)d\psi\left(x+\frac{\psi x\, d\psi x}{dx}\right)}{d\left(x+\frac{\psi x\, d\psi x}{dx}\right)}=\frac{\frac{\psi x\, d\psi x}{dx}\left(dx+d\,.\frac{\psi x\, d\psi x}{dx}\right)}{dx+d\left(\frac{\psi x\, d\psi x}{dx}\right)}=\frac{\psi x\, d\psi y}{dx}$$

From this it appears, that the subnormal is constant in the same series of ordinates, but it does not follow that it must be constant in different series; this property, viz. that $\frac{\psi x\, d\psi x}{dx}$ does not change when x becomes $x+\frac{\psi x\, d\psi x}{dx}$ will furnish us with a solution of the equation in question; for (a) becomes by putting t for $\frac{\psi x\, d\psi x}{dx}$.

$$[\psi(x+t)]^2-[\psi x]^2=t^2$$

where t may be considered as a constant quantity, the general solution of this equation is

$$\psi x=\sqrt{xt+\phi t}$$

ϕt being an arbitrary function of t, therefore the general solution of eq. (a) is

$$\psi x=\sqrt{x\frac{\psi x\, d\psi x}{dx}+\phi\left(\frac{\psi x\, d\psi x}{dx}\right)}$$

or $$y^2=\frac{xy\, dy}{dx}+\phi\left(\frac{y dy}{dx}\right)$$

from which differential equation the curves which satisfy the Problem may be found. It ought, however, to be observed,

that the constant quantity introduced by integration, is not perfectly arbitrary, it must be determined so as to make the equation between x and y fulfil the equation (a). If for instance, we assume $\varphi\left(\frac{ydy}{dx}\right)$ to be equal to $a\frac{ydy}{dx}$, we should find the equation of the curve to be

$$y = (a + x)\,c$$

c being the constant introduced by integration, and on substituting this value of y in (a) we shall find $c = o$, so that

$$y = (a + x)\,o$$

Let us suppose a to be infinite and equal to $\frac{b}{c}$, then we have

$$y = \left(\frac{b}{c} + x\right)c = b + cx = b, \text{ since } c = o$$

which is the equation of a straight line parallel to the axis of the x's, which in fact agrees with the conditions of the Problem. If we suppose $\varphi\left(\frac{ydy}{dx}\right) = a^2 =$ a constant quantity, we should find

$$x = c\sqrt{y^2 - a^2}$$

this value being substituted in (a) gives for determining c the equation

$$c^2\,(c^2 + 1) = o$$

whence $c = o$ and $c = \pm\sqrt{-1}$, using this latter value we have

$$x = \sqrt{-1} \times \sqrt{y^2 - a^2} = \sqrt{a^2 - y^2}$$

which is the equation of the circle, and it is obvious, that this curve satisfies the conditions.

It is very necessary to attend to this mode of determining the constants, as we should otherwise meet in the solution with many curves which do not satisfy the conditions; thus in the last example, the curve is apparently an hyperbola, but owing to the constant becoming imaginary, it is in fact a circle.

To complete the outline of this new method of calculation, it would be necessary to treat of equations involving two or more functional characteristics, and to explain methods of eliminating all but one of them: these lead to a variety of interesting and difficult enquiries, and will probably be of considerable use in completing the solutions of partial differential equations: it would also be proper to consider the maxima and minima of functions, and to apply to this subject the method of variations; these are points of considerable difficulty, and although I have made some little progress in each of them, I shall forbear for the present any farther discussion on this subject. In the mean time, the sketch which I have offered, and the few applications I have given, are sufficient to point out the great importance of this method. It should however be observed, that its applications have only been noticed incidentally; my object has been to direct the attention of the analyst to a new branch of the science, and to point out the manner of treating it: the doctrine of functions is of so general a nature, that it is applicable to every part of mathematical enquiry, and seems eminently qualified to reduce into one regular and uniform system the diversified methods and scattered artifices of the modern analysis; from its comprehensive nature, it is fitted for the systematic arrangement of the science, and from the new and singular relations which it expresses, it is admirably adapted for farther improvements and discoveries.

XII. *Experiments and observations to prove that the beneficial effects of many medicines are produced through the medium of the circulating blood, more particularly that of the colchicum autumnale upon the gout. By Sir* Everard Home, *Bart. V. P. R. S. Communicated by the Society for improving animal chemistry.*

Read March 21, 1816.

A KNOWLEDGE of the readiness with which liquids pass from the stomach into the circulation, carrying along with them the impregnation of different medicines; and the readiness with which such medicines are carried off from the circulating blood, by the action of the kidneys, led Mr. BRANDE and myself to an enquiry respecting the prevention of gravel and gout, upon which subject he has laid two separate papers before the Society.

In these communications, the action of different substances on the contents of the stomach has been considered, and those substances most efficient in depriving them of the principal ingredient met with in stone and gout, are pointed out.

For the cure of gout, the eau medicinale of Husson has been most fortunately discovered to be a specific remedy, and it is now ascertained, by experiments on different people, that a vinous infusion of the colchicum autumnale, or meadow saffron, is equally so, and therefore the two medicines must be considered as the same.

To ascertain their mode of action, appeared to me an enquiry connected with the objects of this Society, which are not con-

fined to the knowledge of purely chemical combinations in the stomach, or other parts of the body, but include the effects of galvanism on the nerves, and of mineral and vegetable solutions on the blood, so far as they affect the actions of life, or the symptoms of disease.

It has already been determined by experiment, that almost every mineral, vegetable, and animal poison, if not the whole of them, is carried into the circulation before it produces its specific effects upon particular parts, whether these are the stomach, skin, or other parts of the body. The most truly specific medicine that we have been hitherto acquainted with, is mercury for the venereal disease, and it is completely established, that this remedy, when in the circulation, is equally efficient in the cure of a recent chancre produced by innoculation, and a venereal sore throat, in consequence of the disease having been carried into the circulation.

That other medicines can be received into the circulation, and, as soon as they arrive there, produce their effects upon different parts of the body, is proved by experiments made by the late Mr. Hunter, although he had no idea of their being usually carried there before they produce the different actions so well known to follow their exhibition by the mouth. He found that infusions of the following substances received into the circulation by the jugular vein, immediately produced the same effects which more slowly follow their being taken by the mouth. Infusion of opium brought on drowsiness. Infusion of ipecacuanha vomiting. Jalap vomiting and purging. Infusion of rhubarb a profuse flow of urine. These effects ceased in a few hours, and appeared to have in no respect injured the animal's health. Except the

venereal disease, gout is the only one whose local symptoms have been completely removed by medicine, in so short a time, as to put it beyond all doubt that their removal is the effect of the medicine. The effect of the eau medicinale and of the vinous infusion of the colchicum autumnale on gout, is indeed more rapid than that of mercury on the venereal disease, but in all other respects corresponds with it, and if these medicines act through the medium of the circulation, the only difference may be, that the one is more quickly received into it than the other.

This power of the eau medicinale, which I have stated to be exactly similar to that of the colchicum autumnale over the local symptoms of gout, I have ascertained by experiment more than six times upon myself; at one time the symptoms went off in six hours, at another in 12, and at others in 24 hours.

As we know the sensible effects of mercury, whether it is introduced into the circulation by the absorbents, or received into the stomach, are the same, we conclude whenever these sensible effects are met with, that mercury is actually in the circulation.

It therefore occurred to me, that if the sensible effects of the infusion of the colchicum should prove to be the same, whether it is introduced into the circulation by the jugular vein, or received by the mouth into the stomach, that we might equally in both cases conclude it to be in the circulation. To determine this point, 30 drops of the vinous infusion of the colchicum (made by macerating two pounds of the fresh roots in 24 ounces of Sherry wine, in a gentle heat for six days, the spirit being previously carried off by heat,) was diluted

with a dram of water, and conveyed into the circulation of a moderately sized dog by the jugular vein. The dog's pulse in a natural state is 140 in a minute.

In 5 minutes, the dog had a tremulous motion of the muscles and fluttering of the pulse, accompanied with nausea, but no retching to vomit. In 14 minutes, the pulse was 180 in a minute and had frequent intermissions. In 4 hours, the pulse was 120 in a minute, of its natural strength, and had frequent intermissions. In 7 hours, the dog had a natural motion, the pulse had no intermission, was 140 in the minute. The dog had a good appetite for food, and appeared in perfect health.

The same dog at the end of three complete days, swallowed 60 drops of the same infusion, exactly double the quantity that had been introduced into the circulation. In 2 hours, he became languid, the pulse wiry and weak, but 140 in the minute. In 4½ hours, the languor much less and the pulse natural. In 8 hours, the dog had had a natural motion. In 11 hours, was in good spirits and very well.

The sensible effects, upon the dog, were similar to those produced upon myself, but in a less degree. Under the influence of a violent fit of the gout, in the ankle, on the 23d of December, 1815, at 10 o'clock in the morning, I took 60 drops of the eau medicinale; the pain of the gout was insufferable, I got into bed, and was so chilly as not to be able to keep my hands warm, even under the bed clothes. In 2 hours, I became rather hot and thirsty. In 3 hours, the pain was so much diminished as to be tolerable, while the limb was at rest. In 7 hours, I had a confined motion from the bowels, and the pain in the ankle became severe, while the foot was placed

on the ground, but this went off as soon as the foot was again placed in a horizontal posture. A nausea, or half sickness, came on; my pulse, which is naturally 80 in a minute, was lowered to 60, and intermitted. In 10 hours, the nausea was gone off, but I remained languid, the pulse beating 70 in a minute. I had some appetite for food.

The following morning, my pulse was 80, and having passed a good night, I was enabled to walk as usual, and follow the duties of my profession.

If these observations shall be confirmed, they must lead us to conclude, that the different kinds of substances, which produce specific diseases, are first carried into the circulation, in the same manner as mineral and animal poisons, and that the medicines by which they are acted upon, go through the same course, before they produce their beneficial effects; a material step will thus be gained in the consideration of diseases, and the modes of treating them.

XIII. *An appendix to a paper on the effects of the colchicum autumnale on gout. By Sir* Everard Home, *Bart. V.P.R.S.*

Read April 25, 1816.

When I laid before the Society my Paper upon this subject, I was anxious to establish what appeared to me to be two important facts; one, that the infusion of the colchicum can be received into the circulation without producing any permanent mischief; the other, that it is through the medium of the circulation, its beneficial effects upon gout are produced, and, therefore, the sudden relief which is experienced can be readily explained. Having attended to the effects of the eau medicinale, and of this medicine for several years in cases of gout, both in my own case, and in those of my friends, I found, invariably, that they diminished the frequency of the pulse, 10 or 20 beats in a minute, and this effect generally took place about twelve hours after the medicine was exhibited: I therefore considered this to be the criterion of the constitution being under the influence of the medicine; and when I found that the pulse was affected in the same way by the medicine received into the circulation, and in a much shorter time, I became satisfied that in both cases this arose from an effect upon the circulation, and not upon the stomach, and therefore did not farther prosecute the enquiry; since exhibiting larger doses could only confirm what is already known, namely, that the medicine is capable, when injudiciously used, of producing very violent effects.

It has been suggested to me since the Paper was read, that the only mode of proving that the medicine acts through the medium of the circulation, is to show that when a sufficient quantity is received into the blood, all the violent effects are produced, that result from a large dose taken by the mouth; and as I had no object but the pursuit of truth, I lost no time in complying with this suggestion, and introduced into the circulation of a dog 160 drops of the same infusion before employed.

The animal instantly lost all power of voluntary motion, the breathing became extremely slow, and the pulse was hardly to be felt. In 10 minutes, the pulse was 84, the inspirations natural, which are 40 in a minute. In 20 minutes, the pulse was 60, the inspirations 30 in a minute, a tremulous motion had taken place in the hind legs. In an hour, the pulse was 115, and irregular; the animal was capable of sitting up, but was in a state of violent tremor, and the inspirations could not be counted.

In $1\frac{1}{2}$ hour, the tremor had gone off, the pulse continued the same; the animal made ineffectual attempts to vomit, and continued to do so for ten minutes, accompanied with great languor; the inspirations were 54 in a minute.

In 2 hours, the pulse was 150, and very weak; the animal had voided $1\frac{1}{2}$ ounce of water, had vomited twice, each time bringing up a quantity of mucus tinged with bile, and had two liquid stools.

In 3 hours, had vomited again, and had another stool; the pulse too weak to be counted.

In 4 hours, continued extremely languid.

In 5 hours, vomited some bloody mucus, and expired.

On opening the body, the stomach contained mucus tinged with blood, and its internal membrane was inflamed; the duodenum had its internal surface universally inflamed, the same appearance in a less degree was met with in the jejunium and ilium, and more strongly marked in the colon than in the ilium.

The facts which I have now adduced, afford sufficient proof of the action of the colchicum autumnale upon the different parts of the body, being through the medium of the circulation, and not in consequence of its immediate effects upon the stomach and intestines.

XIV. *On the cutting diamond.* *By* W. H. Wollaston, *M. D. Sec. R. S.*

Read May 2, 1816.

WHEN we consider how long the diamond has been in common use for the purpose of cutting glass, it is rather surprising that no adequate explanation has been given of that remarkable property, and that even the conditions on which the effect depends have not been duly investigated.

Many persons, indeed, are not aware of the distinction that is to be drawn between scratching and cutting. In the former, the surface is irregularly torn into a rough furrow; in the latter a smooth fissure, or superficial crack, is made, which should be continued without interruption from one end to the other of the line in which the glass is intended to be cut. The skilful workman then applies a small force solely at one extremity of this line, and the crack which he forms is led by the fissure almost with certainty to the other.

Any other substance harder than glass, possesses the power of *scratching* in common with the diamond. But the power of *cutting* has been thought confined to the diamond; and it is true that its peculiar hardness certainly contributes to the duration of that power.

I was informed that persons employed in setting diamonds for the use of the glazier, always select natural diamonds distinctly crystallized, which they term sparks; but upon what

circumstance this supposed superiority of the natural diamond over that which has been cut by art, could depend, I was not able to gain any information.

Having procured a common glazier's diamond ready set, and such a quantity of glass as I thought would be sufficient for learning by experiment the art of cutting, I endeavoured first by forcible pressure on the point in different directions to effect my purpose. But although I could thus tear the surface to a considerable depth, I could by no means command the direction of the fracture.

When I placed the diamond more inclined to the surface, I could occasionally, and in part, obtain what I thought to be a proper cut; but I was unable to continue the stroke with steadiness, and so incapable of repeating it a second time with a similar effect, that I was convinced the precise direction necessary for cutting was confined within very narrow limits.

Having found that the diamond required to be moved in the direction of one of its edges, and having by repeated trials formed a judgment of the requisite inclination of its handle, I mounted it in a frame, in which I could fix it at any angle of elevation that appeared suitable, and could turn it round its axis to adjust the direction of its edge. By this arrangement I had no difficulty in repeating any successful trial, or of varying it according to hints derived from such imperfections as were observable; and I soon discovered that difference in the form of the natural diamond, from that of diamonds cut by art, on which I believe the power of cutting to depend.

When a diamond is formed and polished by the lapidary, all the surfaces are *plain* surfaces, as far as it is in his power to make them so, and consequently the edge or line in which

they meet is straight. But in the natural diamond there is this peculiarity in those modifications of its crystals that are chosen for this purpose, that the surfaces are in general all *curved*, and consequently the meeting of any two of them presents a curvilinear edge. If the diamond be so placed, that the line of the intended cut is a tangent to this edge near to its extremity, and if the two surfaces of the diamond laterally adjacent be equally inclined to the surface of the glass, then the conditions necessary for effecting the cut are complied with. The curvature, however, of the edge is not considerable, and consequently the limits of inclination are very confined; for if the handle be either too much or too little elevated, then one or other extremity of the curve will be made to bear angularly upon the glass, and will plough a ragged groove by pressure of its point. But on the contrary, when the contact is duly formed, a simple fissure is effected as if by lateral pressure of the adjacent surfaces of the diamond directed equally to each side. By that means, adjacent portions at the surface of the glass are forced asunder farther than the mere elasticity of the parts beneath will allow, and a partial separation or superficial crack is produced.

The effects of inequality in the lateral inclination of the faces of the diamond to the surface of the glass, are different according to the degree of inequality. If the difference be very small the cut may still be clean; but as the fissure is then not at right angles to the surface, the subsequent fracture is found inclined accordingly. But when an attempt is made to cut with an inclination that deviates still more from the perpendicular, the glass is found superficially flawed out

on that side to which the greater pressure was directed, and the cut completely fails.

It might be thought that the weakness of the glass in this part would nevertheless occasion it to break in the desired direction; but the bottom of a flaw is in fact of very great breadth when compared to the simple crack produced in a proper cut. In one case the force applied to break the glass is dispersed over a space of some extent, and may be diverted from its course; in the other the whole force is confined successively to the mere points of a mathematical line, which may be conceived the bottom of the fissure, and is directed onward by the facility with which the adhesion of each particle in succession yields to its progress.

The depth to which the fissure made by the diamond penetrates, need not be greater than $\frac{1}{200}$ of an inch, for I found that the fracture might be completely turned from its course, at any part of the intended line, by grinding away a portion of the surface; and by an average of several experiments the thickness of the glass was not found to be diminished so much as $\frac{6}{1000}$ths of an inch.

Since the form of the cutting edge appeared from the above trials to be the principal circumstance on which the property of cutting depends, I thought it not improbable that other stones possessed of the requisite hardness, might be found to produce the same effect, if brought to a similar curvilinear edge. By a little pains I succeeded in giving this form to a sapphire, a ruby, a spinell ruby, to rock crystal, and some other substances, and found that each of these bodies has the power of cutting glass for a short time with

a clean fissure. But notwithstanding the hardness of the ruby was such as to occasion a great deal of labour in giving it the form I wished, the edge of this stone was by no means proportionally lasting. I am inclined to ascribe this defect in part to the grain or position of its laminæ having been unluckily oblique. And it seems highly probable that the singular durability of the edge of the cutting diamond, is owing in some measure to this circumstance, that its hardness in the direction of the natural angle of its crystal, is greater than in any other direction, as we find to be the case in other crystals of which the various degrees of hardness in different directions can be more easily examined.

XV. *An account of the discovery of a mass of native iron in Brasil. By* A. F. Mornay, *Esq. in a letter to* W. H. Wollaston, *M.D. Sec. R. S.*

Read May 16, 1816.

Dear Sir,

Near five years have elapsed since I presented you with a specimen of native iron from Brasil. Particular reasons prevented me at that time, from making it more generally known, and since then my private affairs have not allowed me a moment to look into my notes, and give you this short account of the block from which your specimen was cut, although I have so often promised it you.

In the autumn of 1810, I discovered near Bahia, a spring of water strongly impregnated with iron, which was esteemed a most valuable acquisition in that country. This circumstance called to the recollection of the government, that, about 30 years before, information had been received of the discovery of certain thermal springs, situated at the distance of 40 or 50 leagues to the northward; and as his Royal Highness the Prince Regent of Portugal had enquired, during his stay at Bahia, whether the country possessed any thermal waters, I was requested to visit the spot where they were supposed to exist. The Governor General offered me every facility and protection, and in order to induce me to undertake the journey, some of my friends described to me an

extraordinary *stone* which had been found still farther up the country, in the same direction. It had been supposed to be silver, or iron, or that ferruginous agglomeration so common in Brasil, which often envelopes gold, and I believe sometimes diamonds. On the other hand, some persons who pretended to have seen it, asserted that it was not a mass of any metal, but had only the metallic sound on being struck, common to numerous blocks of stone in the same neighbourhood, called by the inhabitants "serpent stones," in consequence of their exfoliating by decomposition at the surface. As the serpent casts his skin yearly, so they suppose these stones to do.

Some account of the discovery of this extraordinary mass had been given to the government of Bahia, and through the inspector general of the militia, a man of great talents and considerable learning, I obtained a sight of the papers on the subject existing at the government house. On reading them, I was decidedly of opinion, that the mass described was native or meteoric iron, and I determined to go to see it. But before I relate my own observations, I will give you the substance of the notes which I took out of those papers.

In the year 1784, a man of the name of BERNARDINO DA MOTA BOTELHO, while looking after his cattle, noticed the block in question, as being different from all the other stones on the spot, and informed the Governor General of the province of Bahia of his observation. His Excellency immediately ordered the head man of a neighbouring village, that is to say, at the distance of near fifty leagues, to go and examine it. He did so, and reported very marvellous things, calling the mass sometimes iron, and sometimes stone, but giving to understand that it contained gold and silver. The Governor

General commanded him, in consequence, to have it conveyed to Bahia. This man returned to the spot, and after having excavated round the block, so as to be able to get the ends of four powerful levers under it, he contrived by great exertion, with the assistance of thirty men, to turn it on its side. He observed the bed on which it rested, to be of the same scaly substance that was attached to the bottom of the mass, and about eighteen inches thick.

About the latter end of 1785, he conveyed to the spot a waggon, or rather a truck built for the purpose, and succeeded in getting the mass of iron into it, but having spent three days in this operation, the men employed were obliged to depart, in consequence of the neighbouring rivulet being brackish, and not fit to be drank. They returned, however, and yoked oxen to the truck, but they could not move it until they had put on twenty pair of oxen on each side. You must observe that their oxen are not of the strength of ours, that the ground was a loose gravel, and that the truck was constructed on the very worst plan, the wheels being fixed to the axle trees, and the two axle trees remaining constantly in a parallel position with respect to each other.

They proceeded, however, in this manner to the distance of about one hundred yards, when they got into the bed of the rivulet abovementioned, called the Bendegó. There it was stopped by the prominent point of a rock, and as the truck was only calculated to move in a straight line, it was abandoned.

I visited this mass on the 17th of January, 1811, and found it still on the waggon or truck, where it had been lying for five and twenty years. It is situated near the left bank of

the rivulet, but entirely in its bed, which was then dry, and is very seldom otherwise.

I send you a very correct outline of this mass. (Pl. XI.) It is about 7 feet long, 4 feet wide, and 2 feet in thickness, besides a sort of foot on which it now stands, of about six inches in height. The solid contents, however, cannot be inferred correctly from these dimensions, since the broad part is hollowed out underneath very considerably. After making due allowance for the cavities, I estimated, on the spot, the solid contents of the whole mass to be at least 28 cubic feet, which at 500lb. will make its weight to be 14000lb.

Its colour is exactly that of a chesnut, and is glossy at the top and sides, but the hollow part underneath is covered with a crust in thick flakes, outwardly of the colour of rust of iron, and staining the fingers. The flakes are very brittle, and the fresh fracture is black and brilliant, like some magnetic iron ores.

The glossy surfaces of the block are not smooth, but slightly indented all over, as if they had been hammered with a rather large round headed hammer.

There are several cavities in it, from the diameter of a 12lb. cannon ball, to that of a musket ball; the larger ones being shallow, but the others much deeper. They all contain the same substance as is attached to the great cavity underneath, and some of them also fragments of quartzous stones, which I was obliged to break in the holes in order to get them out.

The brown colour of the surface of the block is merely a very thin coat of rust, for the slightest scratch with a knife produces a bright white metallic streak; and yet,

wherever the mass is struck with a steel, it gives out sparks abundantly.

When rubbed with a quartzous pebble in the dark, it becomes beautifully luminous.

The block is magnetic, and even possesses well marked poles. In the outline I have indicated their position. The N. pole is not so well characterized at the shorter point of the same end.

The N. pole of the block lies at present nearly E.N.E.; before it was removed it lay about N.N.E. I ought to tell you that LA MOTA BOTELHO, who first noticed this object, accompanied me, and, as he was present at its removal, he was able to give me much information, being a very intelligent man.

The N. pole is by much the most massive end, and lay deeper in the ground than the other.

No part of the mass has the power of attracting iron filings, whether the spot have been filed to brightness or not.

I had provided myself with a sledge hammer and tools for cutting off some specimens of the iron, but it was with the utmost difficulty that I could detach the few small pieces which you have seen, one of which I gave to you on my arrival in England. The largest I presented to my Lord DUNDAS, to whom I am under many obligations, and who promised to place it in the collection of the Geological Society. I also presented fragments to our lamented friend Mr. TENNANT, and to Dr. MARCET. Another specimen, beautifully crystallized, I disposed of to Mr. HEULAND, and I have only some small pieces left. As soon as the first piece was detached, I was struck with the appearance of internal crystallization not

hitherto noticed in meteoric iron, but as your specimen shows this circumstance very well, I need not describe it.

None of the fragments possess magnetic poles.

No vitreous substance appears about the mass, as in many of the known blocks of meteoric iron.

Having taken a few reagents with me, for the examination of the thermal springs which had been pointed out to me, I tried the malleable part of the mass on the spot, for nickel, and I thought at the time that its presence was indicated; but I am now satisfied that the phænomena which I noticed, might have arisen from iron alone.

I have found my specimens more liable to rust, I think, than wrought iron generally is; and in a damp atmosphere a liquid oozes out from the crevices.

I repaired to the spot where the mass was discovered, namely, on a rising ground on the left bank of the river Bendegó, and caused the soil and gravel to be removed until we came to the bed described in the government documents. We found it at less than three feet depth. I had expected to find in it a considerable protuberance, such as might have fitted the cavity underneath the mass of iron, for I was convinced that the block itself must have been firmly attached to the bed, otherwise it would not have required such a considerable power to turn it on its side.

However, I did not; and thinking that we were not exactly on the spot, I caused two trenches to be opened down to the bed, and crossing each other, the one being between two and three yards long, and the other between one and two. Every part of the bed that was uncovered was perfectly flat and horizontal, except where we dug first; there it was broken, and,

according to the statement of LA MOTA BOTELHO, that was done when the block was removed.

I found no termination to the bed in the directions of the trenches, and at the spot where the mass had laid, it was about one foot thick, or hardly so much; but at one end of the longer trench, not above three inches. I did not break through it any where else. Nearly the same loose gravel appears underneath the bed, as over it. I brought away specimens of the bed, which I considered extremely curious, supposing them to contain nickel. On my return to England I told you, therefore, that I hoped I had found iron ore containing nickel, for I thought that the bed, on which had rested the mass, was one of those of which there are so many all over the province. But as I gave you some specimens, I will not describe it.

The surface of the soil, or rather coarse gravel, at the spot, is about 10 or 15 feet above the main granite rock of the country.

I can only give you an approximation of the latitude and longitude of the place. Th sun was much too high at noon to take its altitude with a sextant and mercurial horizon; and the artificial horizon, which I had been compelled to construct myself, occasioned such a loss of light, as to make it impossible to observe the southern stars for determining the latitude. Different altitudes of the sun at a distance from the meridian, did not give me satisfactory results. I had with me an excellent watch, and having computed the latitude to be about 10° 20′S. I concluded the longitude to be 33′ 15″ W. of Bahia, after making every allowance, and comparing this result with those obtained before and afterwards, at the house of Major DANTAS, called Camuciatá, near Itapicurú.

The rapidity of growth in plants is wonderful in the neighbourhood of the Bendegó, although the main granite rock is so near the surface as to protrude in many places; and what lies on it is chiefly a coarse gravel, consisting of rolled fragments of quartz, felspar and granite of the size of eggs, together with smaller pebbles and sand, which contains, of course, a great deal of mica, but hardly any vegetable earth.

At about 40 leagues to the southward, are found hills of yellow and red sand stone, in which organic remains have not been found; while to the northward, there is a formation of similar hills, in which are observed most beautiful impressions of whole fishes and remains of vegetables.

Between the Bendegó and the sandstone hills to the southward, I observed a great deal of what I certainly take to be basalt. I met with balls from the diameter of two inches to that of upwards of three feet, and numberless prisms, with three and with six faces, scattered about; all of these small, that is to say, about three or four inches in length, and two or three in diameter.

To the southward of the sandstone hills is a sandy plain, almost barren, extending many miles, perhaps 60 or 80, east and west to the sea, but not 20 in breadth, where I crossed it. Small conical hillocks are scattered over it, some of which, the largest, have flat tops, and appear all to be of the same height, about 20 fathoms.

Appearances impressed me with the idea, that they were the remains of a plain which formerly extended over the one on which I then stood, but which had been washed away in a tumultuous manner by a violent current running nearly in an easterly direction. The larger hillocks appear to be

stratified, but they consist of loose sandy materials, except in so far as they contain beds of a dark red iron ore, containing imbedded minute crystals of magnetic iron ore: the thickness of these beds is about two inches, and they are exactly similar to those which are found in the clay hills of Bahia.

The smaller hillocks consist of confused heaps of gravel and loose stones, intermixed with a very large quantity of the same iron ore in fragments, and lumps of manganese, very compact, and of a steel grey colour, containing arsenic, but apparently no iron.

The dreary appearance of this plain is increased by the numerous nests of cupim, (white ants,) standing upright like so many tombstones. On being viewed nearer, they are conical, rather compressed, so that the base is elliptic. All those which I examined were precisely of the same shape. The materials of which they consist are white sand, whitish clay, and particles of wood.

Many of them were full five feet in height.

The soil of the valleys and low grounds, which are occasionally swampy, is abundantly impregnated with sea salt, which the inhabitants wash out for their own consumption; but it contains some bitter salts, which render it purgative to those who are not accustomed to it.

The thermal springs which were pointed out to me, were several, but they hardly deserve the name.

One of them was at 86° of Fahrenheit when the atmosphere was at 81°.

Another was at 88°, when the atmosphere was at $77\frac{1}{2}$°; and also at 88° when the atmosphere was at 80°.

The water of both of these is the purest I had ever seen.

Many small fish were swimming in the basin of the last, from which runs, at all seasons, a considerable rivulet.

A third was at 90° when the atmosphere was at 73°. The water very pure.

A fourth was at 101° when the atmosphere was at $85\frac{1}{2}$°; also at 101° when the atmosphere was at 93°.

Taste of the water rather ferruginous, and very brackish, extremely disagreeable and nauseous. No peculiar smell, and very transparent, although it deposits iron and lime, and an iridescent film is formed on its surface. Contains no sulphuretted gas. The rocks of the neighbourhood contain pyrites not magnetic.

This spring is called the Mai-d'agoa, and is situated on the left bank of the river Itapicurú, near the water's edge, at a short distance from a place called the Mato-do-cipó.

It was during this journey that I had an opportunity of seeing that curious plant called cipó de cunanam. It grows abundantly between Monte Santo and the river Bendegó. It is a climbing plant destitute of leaves; it was so when I saw it, and I believe it to be always the same; it bears no thorns; but often growing so as to form an impenetrable *plica* which the cattle will hardly approach, much less attempt to break through, because when the juice of this plant sticks to their hair, it occasions blisters and great irritation. It contains a milky juice, and I suppose that it is an euphorbium. When I made a cut at the bush with my hanger, in the dusk of the evening, the wounds inflicted presented a beautifully luminous line, which was not transient, but lasted for several seconds, or a quarter of a minute. Having taken a piece of the plant, I bent it in the dark until the skin cracked, when every

crack showed the same light, which is of a phosphorescent appearance. I continued to bend the twig until the milky juice dropped out, when each drop was a drop of fire, very much like what I have seen on dropping inflamed tallow. I did not observe any particular smell. The milky juice is said to be very poisonous; it is caustic, and occasions much itching and irritation when applied to the skin. It becomes viscous in the air, and soon dries of a yellowish colour, slightly tinged with green, when it has the appearance of a gum-resin.

The above account contains all the information that I can give you on the subject: should you think it deserving to be laid before the Royal Society, I would beg of you to add your observations, as they would render the communication interesting.

I am, with sentiments of the highest esteem and respect,

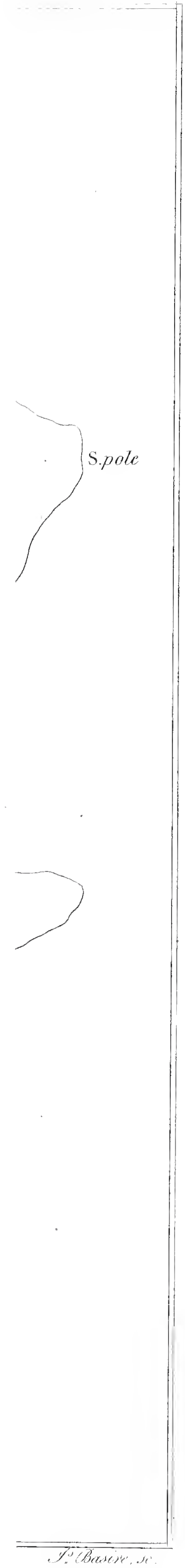
My dear Sir,

Your faithful friend and devoted servant,

A. F. MORNAY.

London, 27th April, 1816.

To Dr. Wollaston,
Secretary of the Royal Society.



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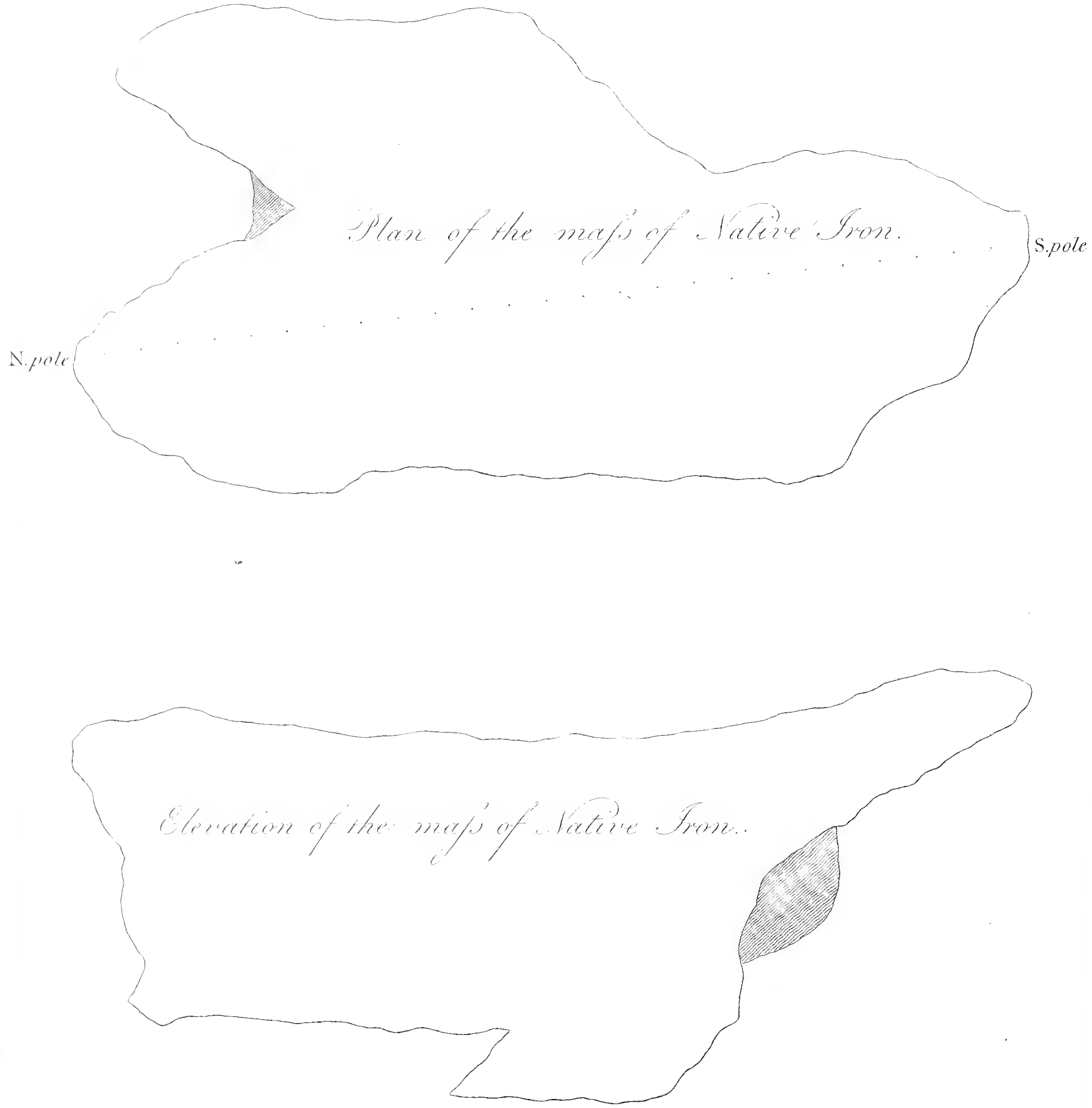
Plan of the mass of Native Iron.
N. pole
S. pole
Elevation of the mass of Native Iron.

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XVI. *Observations and experiments on the mass of native iron found in Brasil.* *By* W. H. Wollaston, *M. D. Sec. R. S.*

Read May 16, 1816.

THE preceding letter from Mr. MORNAY, relating to the discovery of a mass of native iron in Bahia, was drawn up at my request, as a valuable addition to our stock of knowledge on that most curious subject; and I am in hopes that the results of my own experiments may contribute something not uninteresting to the Society.

The specimen of the iron with which Mr. MORNAY very liberally supplied me for experiment, though it necessarily bears marks of the hammer by which it has been detached, presents also other surfaces, not only indicating that its texture is crystalline, but showing also the forms in which it is disposed to break, to be those of the regular octohedron and tetrahedron, or rhomboid, consisting of these forms combined.

In my own specimen, the crystalline surfaces appear to have been the result of a process of oxidation, which has penetrated the mass to a considerable depth in the direction of its laminæ; but in the specimen which is in the possession of the Geological Society, the brilliant surfaces that have been occasioned by forcible separation from the original mass, exhibit also the same configurations, as are usual in the fracture of octohedral crystals, and are found in many simple native metals.

The magnetic qualities of the fragments, fortunately,

enable us to appreciate rightly, those of the entire mass from which they have been detached; for though the mass, when tried upon the spot by Mr. MORNAY, gave indications of having distinct N. and S. poles, it is pretty clear that these were only so by induction, in consequence of position with respect to the magnetic meridian. For though the fragments are not in the least attractive as magnets, and have in themselves no polarity, they are precisely like any other pieces of the best soft iron, and assume polarity instantly, according to the position in which they are held with respect to the magnetic axis of the earth. When a long fragment is held in a vertical position, its lower extremity being then within 20° of the dip of the N. magnetic pole, becomes N., and repels the N. pole of a magnetic needle suspended horizontally. But this power is instantly reversed by being suddenly inverted. So that the apparent contradiction between the observed polarity of the mass, and the seeming want of it in the fragments, is thus completely removed.

Although Mr. MORNAY reasonably expected that this iron would not differ from the many others now on record that have been found in various parts of the world, and from his experiments was led to infer the presence of nickel, it appeared desirable to ascertain this point with more precision than he had been enabled to do, and to determine also in what proportion this peculiar ingredient of meteoric bodies might be found to prevail.

I believe the means which I am accustomed to employ for detecting the *presence* of nickel in native iron to be new, and may deserve to be described, on account of the very small quantity of the iron required for this mode of examination.

Having filed from my specimen as much as I judged sufficient for my purpose, (which need not exceed $\frac{1}{100}$ of a grain), I dissolved it in a drop of nitric acid, and then evaporated the solution to dryness. A drop or two of pure ammonia was then added to the dried residuum, and gently warmed upon it in order to dissolve any nickel that might be present. The transparent part of the fluid was then led by the end of a rod of glass to a small distance from the remaining oxide of iron, and the addition of triple prussiate of potash immediately detected the presence of nickel by the appearance of a milky cloud, which was not discernible by the same means from a similar quantity of common wrought iron tried at the same time.

For the determination of the *quantity* of nickel I employed a different method. A piece of the iron weighing 50 grains having been dissolved in nitro-muriatic acid, the solution was evaporated to dryness. Ammonia was then added, and the solution again evaporated to dryness, in order that the oxide of iron might be rendered more dense, and more easily separated from the soluble portion. A fresh addition of ammonia then readily dissolved the nickel, and the solution after filtration appeared of a deep blue colour.

A small quantity of sulphuric acid having then been added, the whole was again evaporated not merely to dryness, but with sufficient heat to expel the excess of ammonia, muriate of ammonia, and sulphate of ammonia. The remainder was sulphate of nickel which was then redissolved in water, and after being suffered to crystallize weighed 8,6 grains. Having found by experiment previously made for that purpose,

that 10 grains of nickel give 44 grains of sulphate of nickel, I infer that 8,6 of the sulphate correspond to 1,95 of metallic nickel, which is nearly 4 per cent of the quantity of native iron taken for experiment.

By an analysis conducted in a similar manner on 23 grains of the scaly flakes of oxide brought home by Mr. Mornay, from the spot were the mass was found, I obtained 3,1 grains of sulphate of nickel, which correspond to 7,05 nickel, amounting to no more than 3,06 per cent. of the oxidated crust taken for analysis. But, if we consider the weight which 100 parts of the metallic alloy would acquire by oxidation, we shall find the two experiments correspond with a degree of accuracy that may occasion more reliance to be placed on these experiments than they really deserve.

96 parts of iron in the state of black oxide will be combined with 28,3 oxygen
and 4 nickel will take
about 1.1 oxygen,

so that 129,4 of the crust will contain only 4 parts of metallic nickel, and 100 ditto will contain 3,1, which scarcely exceeds the quantity actually found by trial.

From the presence of nickel in this mass we cannot but regard it as having the same meteoric origin with the various other specimens that have before been found; and although in the spot whence it had been first removed, Mr. Mornay discovered a bed of matter from which it appears, by analysis, that similar iron might be formed by art, it seems by far more probable, that an opposite change has really taken place, and that the whole of this supposed ore is the result of pro-

gressive oxidation, during a series of years of which we have no other evidence, and affords the sole ground on which a conjecture could be formed of the very remote period at which this problematic body has fallen upon the earth.

XVII. *On ice found in the bottoms of rivers. By* T. A. Knight, *Esq. F. R. S. In a letter addressed to the Right Hon. Sir Joseph Banks, Bart. G. C. B. P. R. S.*

Read May 23, 1816.

Dear Sir,

Accounts of ice having been found in the bottoms of rivers, have been so numerous, that the existence of it, in such situations, has ceased to be questioned, though no satisfactory hypothesis, relative to the manner in which it can be formed in, or conveyed to, such situations has, I believe, ever been offered. Indeed its existence in such situations does not appear to be a common occurrence, and possibly it has never been seen, during its progressive formation, by any person likely to trouble himself about the causes of its existence. I therefore take the liberty to submit the following account to your consideration.

I first witnessed the existence of ice in the bottom of the water in the river Teme, which passes near my residence in Herefordshire, in the last winter. In a morning which succeeded an intensely cold night, the stones in the rocky bed of the river, appeared to be covered over with frozen matter, which reflected a kind of silvery whiteness, and which, upon examination, I found to consist of numerous frozen spicula crossing each other in every direction, as in snow; but not having any where, except very near the shore, assumed the

state of firm compact ice. The river was not, at this time, frozen over in any part; but the temperature of the water was obviously at the freezing point, for small pieces of ice had every where formed upon it in its more stagnant parts near the shores; and upon a mill pond, just above the shallow streams, (in the bottom of which I had observed the ice,) I noticed millions of little frozen spicula floating upon the water. At the end of this mill pond, the water fell over a low weir, and entered a narrow channel, where its course was obstructed by points of rock and large stones. By these, numerous eddies and gyrations were occasioned, which apparently drew the floating spicula under water; and I found the frozen matter to accumulate much more abundantly upon such parts of the stones as stood opposed to the current, where that was not very rapid, below the little falls, or very rapid parts of the river. I have reason to believe, that it would have accumulated in very large quantities, if the weather had continued sufficiently cold; for I had previously heard, from persons of respectable character, who had no interest, nor, I believe, intention, to deceive me, that, during a long and severe frost, some years ago, before I became an inhabitant of my present house, the whole bed of the river, in the part above-mentioned, had been covered over with a thick coat of ice. But it was not till the month of February that I witnessed the apparent deposition of ice in the manner which I have described; and as the day afterwards became bright, the spicula soon ceased to form, and the ice to accumulate; and before the middle of the day the greater part of it had disappeared.

Upon some large stones, near the shore, of which parts

were out of the water, and upon pieces of native rock, under similar circumstances, the ice beneath the water had acquired a firmer texture, but appeared from its whiteness, to have been first formed of congregated spicula, and to have subsequently frozen into a firm mass, owing to the lower temperature of the stone, or rock. Ice of this kind extended, in a few places, eighteen inches from the shore, and lay three or four inches below the level of the surface of the water, and did not dissolve nearly so rapidly as that which was deposited upon stones more distant from the shores.

The cause of the appearance of large quantities of porous ice, in some of the continental rivers, upon a thaw taking place after a long and severe frost, may, I conceive, be explained, without much difficulty, consistently with the foregoing hypothesis: for such ice would be removed by the increasing force of the rising water, and might be driven together in large masses, provided the temperature of the water were sufficiently low, and that it would be, if afforded by melting snow, or after having flowed over frozen ground. But there have been reports of large quantities of firm and solid ice having been found in this country at the bottom of deep and sluggish rivers, where there existed neither streams nor eddies to occasion the descent of frozen spicula from the surface of the water; and, if such ice have ever been found in such situations, it must be admitted, that it could not possibly have been conveyed there by the means above-mentioned.

I am, &c. &c.

T. A. KNIGHT.

XVIII. *On the action of detached leaves of plants. By* T. A. Knight, *Esq. F. R. S. In a letter addressed to th Right Hon. Sir Joseph Banks, Bart. G. C. B. P. R. S.*

Read June 13, 1816.

Dear Sir,

Since I had last the honour to address a communication to you, with a request that you would lay it before the Royal Society, I have repeated great part of the experiments which formed the subjects of my former Letters, with such additions and variations, as might probably lead to the detection of any erroneous conclusions which I might have drawn: but I have not been able to detect any errors, nor to add any thing very important to my former observations. I have, however, been able to ascertain a few new facts, which I think too interesting to be lost.

I endeavoured, in my former communications, to adduce evidence, that the matter, which becomes vitally united to trees, previously passes through their leaves; and I shall now proceed to state some facts, which, I trust, will prove, that a fluid possessing the power which I have attributed to the true sap, actually descends through the leaf stalks.

A slender knife was passed through some leaf stalks of the vine, about two thirds of an inch distant from their junction to the branch; and, down to that point, the leaf stalks were divided longitudinally, and a transverse section, about half an

inch long, was made through the bark opposite the middle of the leaf stalk. A similar transverse section through the bark, was made somewhat less than an inch distant below; and these sections were united by two longitudinal sections through the bark, which extended from the extremities of the upper transverse sections to the extremities of the lower; by which means, pieces of bark, about half an inch broad, and nearly an inch long, were separated from the adjoining bark. These were then detached from the alburnum, and surrounded by two folds of paper coated with wax on each side; by which all connection and communication with the tree, except through the divided leaf stalks, were cut off. The insulated pieces of bark, nevertheless, continued to grow, and extended downwards, and laterally, and in thickness; and thin layers of alburnum were deposited.

Leaves of the potatoe, without any portion of bark being attached to them, were taken from the plants, just at the period when the tuberous roots began to be formed; and I conceived that these leaves, consistently with my former experiments and conclusions, must contain portions of the living organizable matter, which would subsequently have been found in their tuberous roots. The leaves were, therefore, planted in pots, and placed under glass, where, being regularly and properly supplied with water, they continued to live till winter, though without emitting fibrous roots; and I then expected to find some small tubers at their bases. In this expectation I was disappointed; but the result of the experiment was not less satisfactory, the bases of the leaf-stalks themselves having swollen into conic bodies of more than two inches in circumference, and being found to consist of matter apparently

similar to that which composes the tuberous roots of the plant. The enlarged parts of the leaf-stalks remained alive in the following spring; but whether they are capable of generating buds or not, I have not been able to ascertain.

Leaves of mint were planted in the same manner as those above-mentioned; which grew, and continued alive through the winter, and were still living in the end of the last month, having assumed the character of the thick fleshy leaves of evergreen trees. Upon examining the mould in the pots, I found it to contain very numerous roots, which must have derived their medullary, and their cortical, and alburnous substances from matter which had emanated and descended from the leaves.

I had frequently observed, in former experiments, that the destruction of the mature leaves of young plants not only suspended the growth of the roots, but also the growth of the immature leaves; whence I inferred, in a former communication, that the organizable matter, which composes the young leaves, has always undergone a previous preparation in other leaves of the plant, either of the same, or preceding season; and I was thence led to expect that, under favourable circumstances, the mature leaves might be made to nourish and promote the growth of immature leaves, without the aid of roots. Several shoots of the vine, each about a yard long, were detached from the trees, and laid over a succession of basins of water, into which each of the mature leaves was in part depressed; and thus circumstanced, the young leaves continued to grow, and the points of the shoots to elongate; and all were alive, and in perfect apparent health at the end of a month. The water necessary to preserve the young leaves

must in this case have been derived from the mature leaves; and I entertain no doubt, but that the organizable matter which occasioned their growth, was derived from the same source. Intersection of the bark between the mature and young leaves was not attended with any injurious consequences, and the sap must, therefore, have passed to the young leaves through the alburnum.

Consistently with the preceding circumstances, if the mature leaves be destroyed, or taken off, the fruit ceases to grow, or, if full grown, remains without richness or flavour; and the power of feeding fruits in winter and early spring seems to be confined to evergreen plants. The orange and lemon tree, the ivy and holly, afford familiar examples of this; and where a genus of plants consists of evergreen and deciduous species, as that of mespilus and viburnum, the evergreen species alone nourish their fruit in winter and early spring.

The probable passage of the sap from the mature to the young leaves and fruit, may, I think, be easily pointed out, though decisive proof of its course will probably never be adduced. Having often detached the bark from the alburnum of the stems of young oaks, just at the period when the midsummer shoots were beginning to elongate, I observed, as others have done, that a fluid exuded from those parts of the surface of the alburnum, which are called (most improperly) the medullary processes, and from correspondent points of the bark, which resemble the medullary processes in organization. This fluid has been proved, by its power of rapidly generating an organic substance, to be the true sap of the tree, part of which, I conceive, at this period, to be passing from the bark to join the ascending current in the albur-

num; which current feeds the young succulent shoots and growing leaves. Subjecting the alburnum to a slight degree of pressure at this period, I found that a considerable quantity of liquid, being apparently the true sap of the tree, issued out laterally through the medullary processes, as well as longitudinally through the cellular substance of the alburnum: but the tubes of it continued empty, and their position was marked by depressions of the surface of the extravasated fluid. I endeavoured to ascertain, what proportion of water a given quantity of the alburnum of such oak trees contained at this period; and I found that 1000 parts lost by drying only 371 parts: which is not more than the weight of the water that the cellular substance appears capable of containing, entirely independent of the tubes. That the tubes, nevertheless, are not always empty, but that they act at other periods of the year as reservoirs for the sap, I have given an opinion in a former communication; and I am now in possession of facts which prove them to perform this office, even in the heart wood, to a much greater extent than I had ever at any former period suspected; and which incline me to believe, that the durability of the heart wood, as well as of the alburnum of the oak, will be found to depend to a great extent upon the period in which the tree is felled: but I propose to make my observations upon these points the subject of a future communication.

I am, my dear Sir, &c.

T. A. KNIGHT.

XIX. *On the manufacture of the sulphate of magnesia at Monte della Guardia, near Genoa.* *By* H. Holland, *M. D. F. R. S.*

Read June 13, 1816.

The following account, which I have the honour to present to the Royal Society, of the manufacture of sulphate of magnesia at Monte della Guardia, near Genoa, has been drawn up chiefly from the note I made, when visiting this spot, in the spring of 1815. These notes have received additions from a paper, drawn up in 1803, by my friend Signore G. Mojon, lecturer on chemistry at the college of Genoa, who has bestowed much attention on the subject, and made several experiments on the composition and comparative purity of the manufactured salt.*

The Monte della Guardia, situated eight miles to the N. west of the city of Genoa, is one of the higher points in that part of the chain of the Ligurian Appenines, which borders immediately on the coast. The mountains in this portion of the chain are chiefly of primitive slate, with some subordinate formations of marble and serpentine, a considerable extent of transition and secondary limestone, chiefly to the eastward of Genoa, and a few more partial deposits of sandstone and coal.

The summit of Monte della Guardia is somewhat more than 2000 feet above the level of the sea, from which it is five

* This paper was published by the Society of Medical Emulation in Genoa.

miles distant. The ascent towards the mountain from Sestri, the nearest point on the coast, is uniform, though rugged; following the course of the rapid torrent, called the Panigaro. The ravine of this torrent is interesting, inasmuch as it forms the division between a high ridge of serpentine, which occurs on its eastern side; and an insulated formation of magnesian limestone, which composes the hill of Monte del Gazzo, to the north of Sestri, and other adjacent heights on the western side of the stream. This magnesian limestone, which in Monte del Gazzo reaches a height of more than 1200 feet, forms in various places naked cliffs, distinctly showing the stratification of the rock; which stratification, as well as the colour of the limestone, is very remarkably contrasted with the character of the serpentine rocks, on the opposite side of the ravine.

The latter, together with some argillaceous and chlorite slate of the same formation, compose for several miles, the ridge of hills between the torrent of Panigaro, and the broader valley of the Polcevera to the east; stretching upwards in a direct line from the sea towards the summit of Monte della Guardia. The primitive schistus is doubtless the base of this formation; the serpentine reposes upon it in vast unconformable masses; exhibiting a surface of great ruggedness, and almost entirely bare of vegetation; with the dark green colour, and usual resinous lustre of this mineral. I observed much steatite, talc, and asbestus in it; also many small veins of pyrites. The asbestus of this place affords numerous fine specimens to the cabinets of Genoa.

Monte della Guardia may be considered the summit of this ridge. It exposes to the south a somewhat concave outline;

the front of which is deeply worn out, so as to show the stratification of the primitive slate, mixed with chlorite slate, with the other magnesian minerals just noticed, and with numerous veins, or layers of copper and iron pyrites. The specimens of this pyrites have a steel grey, or greenish yellow colour, according to the respective predominance of iron or copper. The whole substance of the pyrites has a schistose structure, corresponding with that of the rocks in which it is situated. It is so intimately mixed with the same magnesian minerals, as to be for the most part extremely unctuous to the touch. It is in general easily sectile. The specific gravity varies from 3.6 to 4.6.

It is on this side of the Monte della Guardia, and from the materials just described, together with the magnesian limestone of Monte del Gazzo, that the manufacture of sulphate of magnesia has been established; it is at present carried forward on a small scale, but is evidently capable of extension and improvement. The original object of attention in this place was the working of the copper and iron pyrites; but the observation of the crystals of sulphate of magnesia, formed during the processes applied to these ores, changed in great degree the character of the manufacture; and the green vitriol and copper obtained here are now become secondary objects to the proprietors. The process employed for the formation and separation of the sulphate of magnesia is briefly as follows: the pyrites is worked out of the mountain by tunnels, which the steep natural section of its front, and the general direction of the veins, allow to be carried nearly on an horizontal level. Some of these galleries, which I saw, were more than 200 feet in length, and varying from 10 to 15 feet

in width. Others, which are now filled with water, or destroyed by the crumbling down of the rock, were described to me as of still greater length. The ore, thus obtained, is broken down into small pieces, and roasted in an open pit, or kiln, about 20 feet in depth; with the alternation of wood, so disposed as most effectually to aid the combustion. This process of roasting continues generally for ten days. The kiln being sufficiently cooled, the ore is removed thence, and disposed in large heaps, underneath a shed in the vicinity. Here it remains for several months fully exposed to the air, and occasionally moistened with water thrown upon it, to aid the chemical changes by which the salt is found. An efflorescence of sulphate of magnesia soon commences, and gradually proceeds, so as to cover with minute crystals the surface of the ore, which, during this time, crumbles down into very small fragments.

This is the usual commencement of the process, but there are varieties of the ore, more easily decomposable, which require less roasting, or may even be submitted to the after processes, without any application of heat. Other varieties, on the contrary, require more roasting, to prepare them for the proper efflorescence of the salt, which efflorescence itself varies in rapidity, according to the particular description of the ore.

The materials, brought into this strata, are then lixiviated; and the liquor containing in solution the sulphates of magnesia, of copper, and of iron, is filtered; in passing it through layers of sand and straw, disposed in large wooden vessels. These processes repeated as often as may be necessary, the next object is the separation from the liquor, of the metallic

sulphates. When the sulphate of copper is perceived to be abundant, the metal is precipitated by refuse of iron, introduced into the liquor. The sulphate of iron, and any remaining sulphate of copper, are decomposed by the addition of lime, in the state of milk of lime, which causes a precipitate of sulphate of lime, together with the metallic oxides.

The lime employed for this decomposition, is obtained from the magnesian limestone, already described as belonging to this district. That of Monte del Gazzo is preferred, both as nearer to the spot, and possibly from its containing a larger proportion of magnesia, than the other magnesian limestones of this coast, about 16 per cent., as appears from analysis made of it. It appears probable, that a certain portion of sulphate of magnesia is obtained from the use of this limestone, in decomposing the metallic salts, and the process of manufacture thereby rendered somewhat more productive. The quantity of lime employed may equal about $\frac{1}{100}$ part of the weight of the pyritic ore.

The metals being thus separated from the saline liquor, it is filtered anew, and then evaporated to a certain point in a large copper boiler. It is subsequently let off into small glazed earthen vessels where the crystallization of the salt takes place. M. Mojon mentions the circumstance of its being needful frequently to disturb the crystallization, to satisfy the prejudice of the purchasers, who consider it essential that the salt should be delivered to them in the form of small spicular crystals.

The produce of sulphate of magnesia varies of course according to the quality of the ore. In general, perhaps, it may be stated at $\frac{1}{10}$ of the weight of the material employed. The

ore remaining after the first filtration, is usually roasted and lixiviated a second time, to obtain the portion of salt which it is still capable of affording.

It is almost unnecessary to make any remarks on the theory of the process just described. The sulphuric acid, formed by the action of heat, air, and water upon the pyritic ore, combines with the metals, and with the magnesian earth, which the ore contains. I am not aware of any experiments that have been made, to ascertain the proportion of salt which may be derived from the use of the magnesian limestone. Probably, however, it is extremely small; and it would seem very doubtful, from the nature of the process, whether any improvement to the manufacture can be expected from this source. The effect of employing more of the limestone than is strictly necessary, would be, that the lime, rather than the magnesia, would unite with the sulphuric acid; and if the addition of the mixed earth were in excess, the lime alone would have effect in the decomposition; and sulphate of lime, with the metallic oxides, would be the new products obtained.

M. Mojon has proposed a variation in the manufacture at Monte della Guardia; depending on the different tendency of the sulphate of magnesia, and of the metallic sulphates towards crystallization. The principal object of the variation would be to save the sulphate of iron, now lost by decomposition; and to increase the value of the produce of copper by obtaining it in the state of sulphate. This change in the process has not hitherto been adopted; and probably might be found to be attended with several difficulties in practice.

The buildings connected with the establishment at Monte

della Guardia, are on a small scale; their situation is a singular one, at the height of 1600 feet above the level of the sea. The present conductor, and, as I believe, the chief proprietor of the establishment, is Signore Ansaldo of Sestri, a man of intelligence and activity.

The produce of the sulphate of magnesia, as far as I could ascertain from enquiries on the spot, does not at present exceed $1\frac{1}{2}$ cwt. per week. It might doubtless be increased to a much larger amount, from the abundance of the materials of manufacture. The colour and general appearance of the salt are good, and M. Mojon has found it by analysis to be a very pure sulphate of magnesia. It is used pretty extensively in the medical practice of Italy, under the name of the Sal Inglese; and might, in the extension of the manufacture, become an article of very considerable profit to the proprietors of the establishment.

Before concluding this paper, I may add, that I have been informed of an analysis by Vauquelin of the pyritic ore of the Monte della Guardia; but, from the want of sufficient references, I have hitherto been unable to ascertain the results of this examination.

XX. *On the formation of fat in the intestine of the tadpole, and on the use of the yelk in the formation of the embryo in the egg. By Sir* Everard Home, *Bart. V.P.R.S.*

Read May 23, 1816.

THE tadpoles in England are so small, that no person has attempted to make an accurate investigation of the internal changes of structure that take place in them, between the time of the animal leaving the egg and becoming a frog.

The rana paradoxa of Surinam, in its tadpole state, is larger than in any other species of frog yet known, and so closely resembles fishes, that in that country it is sold as such in the market, for the use of the table, under the name of Jackie; and, as the frog produced from it, is in the first instance as large as a common frog in this country, it is highly probable, that it becomes an animal of considerable size. Mr. IRELAND, a surgeon in the army, who resided several years at Surinam, and watched the tadpole through many of its changes, has brought home specimens in different stages of its metamorphosis. These he has kindly submitted to my examination, and has in the most liberal manner, deposited the specimens in the Museum of the Royal College of Surgeons in London, of which he is a member.

As Mr. IRELAND had no opportunity of examining the tadpole, before the hind legs begin to make their appearance through the skin; with a view to render the series complete, I have since examined the progress of the changes, from the

ovum in the spawn of the English frog.* The jelly of which the ova are composed, I have upon a former occasion given an account of to the Society. The ova themselves differ from those of snakes and lacertæ in general, in having no yelks. When the tadpole is once formed, it appears to feed upon the jelly, which although not absolutely albumen, is a near approach to it.

In this stage, each ovum is pressed into the form of an hexagonal prism with flattened ends, so as to form the whole into one compact mass.

The tadpole, after it leaves the ovum, has on each side ten filaments projecting from the neck, for the purpose of aeration of the blood; such filaments must be considered as temporary gills.

The lacerta of this country, called the newt or eft, in its larva state, has the same projecting filaments, which drop off when the gills are formed; they are more complex in their structure, and only three in number, on each side. This circumstance shows, that the larva of an eft is a species of tadpole, and that the eft itself does not belong to the tribe of lizards, but is a nearer approach to that of frogs. In the tadpole, as soon as the abdomen begins to enlarge, these external filaments disappear. Twenty-four similar filaments are met with in the fœtus of the shark while contained in the egg, which drop off before the fœtus escapes from the shell.

The spawn of the English frog was collected on the 1st of April, 1816. On the 15th, the tadpole left the egg, but the filaments or external gills were not visible, only a deep notch

* The tadpoles having become frogs since this Paper was read, I have been enabled to complete the series.

on each side nearly separating the head from the body. On the 23d, the ten filaments on each side were distinct; on the 27th they disappeared. In June, the external orifice on the left side, for the water to pass off from the gills, was very distinct, but none was seen on the right. On July the 8th, the hind legs began to appear, but the toes were not separated. On the 14th of July, the hind legs were seen externally completely formed, and on opening the skin of the chest, the fore legs were equally so; but there was no external projection by which this could be known. The lungs were completely formed. On removing the intestine, there was no fat deposited on the loins. On the 16th, the contents of the intestine were voided in considerable quantity. On the 18th, the elbows of the fore legs projected under the external skin, and so much of the contents of the intestine had been voided as to give a taper form to the lower part of the body. On the 19th, the fore legs were completely disengaged and appeared externally; the mouth had become wide like that of a frog. The tail had a notch at that part where it afterwards separates; the intestine was reduced in diameter, and to the length of that of a frog; an appearance of oil was seen on the loins. On the 23d, the tail had dropped off, leaving the projecting root. The animal had left the water and remained among the grass. Behind the intestines upon the loins were several small membranous appendages in an empty state.

On the 28th, the root of the tail had wholly disappeared; the appendages had become more opaque.

The ova of the frog appear to be hatched at very different periods, since some of the tadpoles become complete frogs, before others have their hind legs protruded through the skin.

Upon examining the tadpole of the rana paradoxa, just

when the hind feet appear externally, I found the mouth very small, and nearly round, the teeth cuticular, the upper ones overlapping the under, the œsophagus, stomach, and intestine, forming one uniformly continued canal, which passed down to the lower part of the abdomen; it was bent upon itself, passed up again, and then made a great number of coils in a circular form; its coats were very firm, its capacity very small. There were three gills completely enclosed on each side, and a little way below the eye on the left side, a small round orifice, for the water, by which the gills are supplied, to pass out; but none on the right. When the tadpole is arrived at its full growth, and the hind legs are completely formed, which takes place, according to Mr. IRELAND'S observations, in 14 days after their first appearance, the cavity of the abdomen had become exceedingly enlarged, the intestine very capacious, its coats almost as thin as cobweb: it was completely distended, through its whole extent, with a soft substance, which when burnt had the smell of hay. Behind the intestine, all along the posterior part of the abdomen, a large quantity of fat was met with of a yellow colour, enclosed in long, thin, transparent membranous bags; no part of this fat was met with in the prior stages of the tadpole's growth. The lungs were completely formed.

When the mouth of the tadpole has been changed into that of the frog, and the fore legs completely protruded, but the tail remaining entire, which happens 21 days after the last mentioned change, the large coils of intestine were found contracted into a canal one fourth of its original length; the coats had become as firm as those of an artery, the external surface was corrugated and the canal empty. The stomach had become a distinct cavity, and there was a contraction, where it

terminates in the intestine. All these parts were embedded in fat, which filled every part of the abdomen, not occupied by the liver, which had acquired a large size. The lungs were filled with air, and the gills had entirely disappeared.

When the tail has dropped off, leaving the projecting root, which takes place in seven days more, the only internal change met with, was, that no fat whatever was found in the cavity of the abdomen.

The great length of the intestine which has been described, has nothing analogous to it in the caterpillar, and is probably confined to the frog tribe.

The egg of a frog bears no proportion in size, to those of the other animals of the same class, and differs from them in having no yelk, therefore, although it contains sufficient materials for the formation of the tadpole, something is still wanting, before it can be metamorphosed into a frog; and in the tadpole state, a store of fat is laid up, beyond what is required for its own immediate support and future growth, to furnish the necessary means of supplying the different structures in the frog, not already existing in the tadpole; and this fat appears to be formed in the intestine.

The length of intestine in the tadpole, when its relative proportion to the size of the animal is considered, exceeds every thing of the kind that is met with in other animals.

In the tadpole of the Surinam frog, the intestine after it has acquired its full size, does not remain of this enormous length, beyond the period of its metamorphosis into the frog taking place; and what is deserving of particular attention, the fat is deposited, when the intestine has acquired its full size, and no sooner is the intestine reduced in length, than not only no

more fat is deposited, but all that was previously formed is found to have been consumed, in producing the metamorphosis into the frog: which leads me to conclude, that such a deposit of fat is necessary to the metamorphosis of a tadpole into a frog, and that such unusual length of intestine, is required to admit of so large a quantity of fat being formed in so short a time, and, therefore, that the intestine is the laboratory in which the fat is formed.

To ascertain, whether the necessity of such a supply of fat is occasioned by the soft parts of the tadpole not being convertible into bones, and other parts of the frog, which did not exist in the tadpole, or, simply, from a deficiency of materials, I have had the assistance of my friend and fellow labourer in animal chemistry, Mr. HATCHETT, who some years ago ascertained, that the yelk of an egg is essentially composed of concrete oil, combined with a small proportion of albumen, and he has made out the following important facts. "That the spawn of the frog, has no yelk, and contains no oil whatever; he also corroborates Mr. BRANDE's statement, that it consists of a substance intermediate between albumen and gelatine, inclining principally to the former. That the ova of the shell snail, both of those that have a shell, and those that have only a strong membranous covering, have no yelks, and consist of albumen, since they coagulate in proof spirit of wine, and, when so coagulated, and examined some time afterwards, appear not to contain any oil. That the ova of the lobster have no yelk, and contain no oil.

But he remarks, that the spawn of the lobster, when recent, is filled with albumen mingled with a substance of a dark olive colour, and whilst the former as usual is coagulated by heat,

the latter becomes of a vivid red; this, Mr. HATCHETT observes, is the colouring matter of the shell, which three or four years ago, Mr. BRANDE found to become red by the application of acids without heat; for dilute sulphuric, nitric, muriatic, and the strong acetic acids immediately produce the same effect on the colouring matter of the spawn, but this is not the case when a weak acid, such as common distilled vinegar, is employed.

When this bright red colour has been produced by the above acids, it appears to be permanent, excepting when nitric acid has been used, for then the red colour changes to yellow, which by the affusion of ammonia becomes orange colour, as is usual with animal substances so treated.

Dilute nitric acid in which it had been digested, afforded slight traces of a phosphate, which was not phosphate of lime.

As the red colour is produced by acids as well as by heat, there was some reason to expect that it would have been destroyed, or at least that its intensity would have been diminished by a great excess of the alkalies, but not the smallest effect was produced by any of these, and indeed so far from it, that the recent spawn when put into a solution of caustic potash, became in a few seconds changed to as bright a red as when the mineral acids had been employed.

The red colour is also produced by the effects of air, light, and the evaporation of moisture, for paper or linen which have been stained with the olive coloured substance, become red in the course of a few minutes, so that in this respect it somewhat resembles the secretion obtained from the buccinum lapillus, or purple whelk. The purple colour of this last does not, however, suffer any change, whilst the colouring

matter of the lobster in the course of some days becomes of an ochraceous colour. In this state it seems to be permanent, for it was retained by linen which had been marked with it, after repeated boiling in water and washing with soap.

From these cursory experiments, Mr. Hatchett observes, that this animal colouring substance is apparently of a peculiar nature, and that it is the same in the common cray fish and the prawn, as well as in the sea cray fish and the crab, but in the two last, it has already assumed its red colour.

That the ova of the salmon and pike have no yelk, and consist principally of albumen, as they coagulate by heat, but contain also a small portion of oil, which perhaps is a substitute for yelk.

That the ova of the cartilaginous fishes, as well as those of the lizard and snake, have a regularly formed yelk, like that of the hen, composed of the same ingredients; but in both the viviparous and oviparous sharks there is no perfectly formed albumen, but in its place a gelatinous substance, which Mr. Brande ascertained to be intermediate between gelatin and albumen, similar to what is met with in the spawn of the frog.

In addition to that which has been above stated, Mr. Hatchett has communicated to me the following observations.

"The yelk of the eggs of birds is principally and essentially composed of a butyraceous oil, combined with a small proportion of albumen, the average of which in the yelk of the common fowl amounts to about one fifth. Yelk, when triturated and diluted with water, forms (as is well known) an emulsion, and yelk may be regarded as an emulsion in a high state of concentration.

In milk, the caseous part, or curd, corresponds to the albuminous part of yelk, as the butter in milk does to the other part or oil of the yelk. The principal difference, therefore, between milk and yelk is, that the former is in a dilute, and the latter in a concentrated state. Hence, Mr. HATCHETT observes, it appears that many of the oviparous animals during the period of incubation, are nourished by a pabulum similar in quality to that by which young viviparous animals are supported, whilst the great degree of concentration of this pabulum in the first case is essentially necessary, in order that the quantity of nutritious matter which is required during incubation, and which is included with the animal within the egg, should be condensed into the smallest possible bulk.

Young viviparous animals are at first incapable of supporting themselves by those substances which are afterwards to become their food, and they are therefore nourished for a certain period of time by the milk of their mothers; but young oviparous animals, such as the chicken, partridge, and birds in general, come forth from the shells complete in their bodily faculties, and immediately partake of the food to which the parent birds are accustomed, so that it seems they are prepared for this, and are nourished during incubation, by a substance similar in its nature to that by which young viviparous animals are supported, or suckled, during a certain time after their birth, and that the process corresponding to that of suckling, is, with regard to birds, performed and completed in the course of incubation.

The experiments which Mr. HATCHETT has made upon the ova of these different tribes of animals, lead to the conclusion, that in all ova, the embryos of which have bones, there is a

certain portion of oil, and in those ova whose embryos consist entirely of soft parts, there is none.

This conclusion is much strengthened by the peculiarity, which it has been my object in this paper to point out, of the tadpole laying up a magazine of fat before the metamorphosis into a frog takes place; it is, therefore, rendered probable, that a certain portion of oil is necessary for the formation of bone, and that the proportion in different ova corresponds with the greater or less degree of hardness of the bones of the fœtus.

XXI. *On the structure of the crystalline lens in fishes and quadrupeds, as ascertained by its action on polarised light.* By David Brewster, *LL.D. F. R. S. Lond. and Edin. In a Letter addressed to the Right Hon. Sir Joseph Banks, Bart. G. C. B. P. R. S.*

Read June 20, 1816.

DEAR SIR,

THERE is, perhaps, no subject in natural history which has excited so much attention, as the structure and functions of the eyes of animals; and there is certainly none which has so ill repaid the anxiety and labour with which it has been investigated. The physiologist was naturally led to study the mechanism of an organ through which man receives the noblest of all his enjoyments; and the natural philosopher, considering it as the work of infinite intelligence, ardently anticipated the improvement of optical instruments from the imitation of this perfect model. It is discouraging however, to estimate the real amount of the labours of the one, and to perceive how little advantage has been derived from them by the other. The most prominent functions of the eye are still very imperfectly understood, and the improvement of the telescope has been retarded, rather than advanced, by the pursuit of a false analogy. I have, therefore, some satisfaction in being able to throw additional light upon a subject of such difficult investigation, and so generally interesting both from its optical and physiological relations.

Having found that the doubly refracting structure could be communicated to glass and other bodies, by giving them a variable density either through the agency of heat or mechanical pressure, I was led to conclude, that the same structure would be found in the crystalline lens of fishes and other animals, which was known, by direct experiment, to increase in density towards the centre. I had formerly examined the action of the crystalline upon polarised light, without obtaining any new result; but I now placed such reliance on the truth of the general principle, that I resumed the subject with the utmost confidence of success.

Upon exposing to a polarised ray the crystalline lens of a large cod, included in its capsule, I could not perceive, as happened in my early experiments, any very distinct indications of a peculiar action. I plunged it, however, in Canada balsam contained in a hollow parallelopiped of glass, and was surprised to observe a regular optical figure varying its shape during the revolution of the crystalline.

I now turned the spherical crystalline, till the diameter which corresponded to the axis of the eye, or the line joining the poles to which the fibres converge, was parallel to the polarised ray, and I observed the appearance shown in Fig. 1. (Pl. XII.) consisting of twelve luminous sectors, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, separated from each other by a black cross, and by two dark concentric circles. The interior sectors, 1, 2, 7, 8, were small and exhibited a white tint of the first order, increasing in brilliancy towards the centre. The middle sectors 3, 4, 9, 10, which are very large, are separated from the interior ones, by a broad dark circle, and display a white tint of the same intensity. The outer sectors

5, 6, 11, 12, are extremely faint, and are seen with considerable difficulty in this position of the lens.

If the crystalline is now turned round, so that its axis, which corresponds to the axis of vision, may always be parallel to the polarised ray, the same appearances will be seen without the slightest variation. But if this axis is inclined to the polarised ray in the direction 1, 2, the sectors 1, 2, will diminish, and 7, 8, will increase in size, and an additional white spot will appear at the centre as in Fig. 2, till by increasing the inclination, the sectors 1, 2, and the white spot will completely disappear, leaving the sectors 7, 8, much enlarged and of a bluish white tint. If the inclination is in the direction 7, 8, the sectors 1, 2, will increase, and 7, 8, will diminish in the same manner.

By transmitting the polarised light through other two faces of the glass parallelopiped, so as to traverse the crystalline in a line perpendicular to its axis, the optical figure presented new appearances. When the axis of the lens was either parallel or perpendicular to the plane of primitive polarisation, which happened four times in the course of a revolution, it exhibited the form shown in Fig. 3. The tints 1, 2, 7, 8, were now reduced to a pale blue of the first order, and the black cross was very ill defined at the centre. The middle sectors 3, 4, 9, 10, were a little reduced in size, while the exterior ones 5, 6, 11, 12, had experienced a very sensible augmentation. At intermediate positions of the crystalline, when its axis was inclined 45°, 135°, 225°, and 315° to the plane of primitive polarisation, the optical figure assumed the appearance shown in Fig. 4, where the two sectors 7, 8, are greatly enlarged, and the other two 1, 2, have wholly, or almost wholly, disappeared.

I now removed the capsule of the lens, so as to let out the semifluid matter which it enclosed, and having rubbed off the very soft exterior coat, I immersed the diminished sphere in Canada balsam, but I could never observe the exterior sectors, 5, 6, 11, 12, all the other appearances being exhibited as before. I next removed the middle coats of the crystalline, and replacing the nucleus, which was now reduced to one-eighth of an inch in diameter, in the glass parallelopiped, I observed the central sectors 1, 2, 7, 8, without any of the middle or exterior ones. By pressing the nucleus between two plates of glass, or by allowing it to indurate gradually, the depolarised tints ascended in the scale of colours, as in the case of animal jellies.

If we now take a plate of sulphate of lime which polarises a blue of the second order, and combine it with the crystalline lens, so that its axis may be parallel to the line 6, 10, 2, 1, 9, 5, the white tints 9, 10, will ascend to a green of the second order, while those at 1, 2, 5, 6, will descend to an orange red of the first order. In like manner, if the axis of the plate of sulphate of lime is parallel to the line 11, 3, 7, 8, 4, 12, the tints 3, 4, will become green, and 7, 8, 11, 12, an orange red.

Hence, it follows, *that the nucleus* 1, 2, 7, 8, *and the exterior coat*, 5, 6, 11, 12, *have the same structure as one class of doubly refracting crystals,* while the middle coats 3, 4, 9, 10, have the structure of the other class.

In order to compare these different structures with those of glass crystallized by heat and by pressure,* I took a polished sphere of crown glass, and, having brought it to a red heat, I cooled it by rolling it quickly in every direction over a smooth surface. When it was immersed in Canada balsam,

* See Phil. Trans. 1816, p. 46, 156.

and exposed to polarised light, it had the appearance shown in Fig. 5, in whatever position it was held, the highest tint being an orange yellow of the first order. By examining these sectors with sulphate of lime, I found that the glass had the same structure as the middle coats of the crystalline. In like manner, it appeared that the sectors, exhibited by pressing a convex lens upon a flat piece of glass, were produced by a structure the same as that of the central nucleus of the crystalline. The structure of the crystalline lens, in short, is similar to that of a plate of glass that gives the unusual fringes* bent into a circular shape. Hence it follows, *that the central nucleus and the external coat are in a state of dilatation, while the intermediate coats are in a state of contraction, and that these opposite states are not dependent upon each other as in crystallized glass.*

The phenomena which have now been described are visible also in the crystalline of the *haddock.* They appear likewise in that of *sheep* and *oxen,* but there is here only one series of luminous sectors corresponding with the intermediate set in the crystalline of fishes. The human crystalline will no doubt display similar properties, but in an inferior degree.

The *cornea* both of fishes and quadrupeds, and also the human cornea, have an analogous structure; in which the optical axes of all the particles are directed to its centre. Its structure is the same as that of the internal nucleus, and it produces an effect upon polarised light similar to what is shown in Fig. 6.

The *sclerotic coat* of fishes has the property of depolarising light in separate spots like the diamond, or a mass of crushed

† See Phil. Trans. 1816, p. 65, 66.

isinglass; but it derives this property from a bluish white membrane which covers the outside of it, for when this is removed, it loses the doubly refracting structure. If the sclerotic coat is boiled, it is capable of receiving the structure of doubly refracting crystals by mechanical compression and dilatation. In its natural state, it possesses the same property, but in an inferior degree. The cornea is also capable of having its doubly refracting force increased by compression or dilatation.

From these experiments the following conclusions may be deduced.

I. All the parts of the crystalline lens of fishes corresponding to the two dark concentric circles, exercise no action upon polarised light. The outward spherical shell which acts upon light like one class of doubly refracting crystals, and also the solid nucleus which exercises a similar action, are in a state of mechanical dilatation, while the middle spherical shell which acts upon light like the other class of crystals, is in a state of mechanical contraction.*

II. The structure of the crystalline lens in fishes is not symmetrical, as has hitherto been supposed, consisting merely of a number of coats of different densities; but it has a distinct relation to that diameter of the sphere which is the axis of vision.

III. The variations of density which produce the doubly refracting structure, are not related to the centre of the crys-

* When the crystalline lens is examined by common light, there is an obvious appearance of a rapid change of density at the line which separates the middle and the exterior sectors. This is probably the boundary of the fluid coat adjacent to the capsule.

Fig. 1.

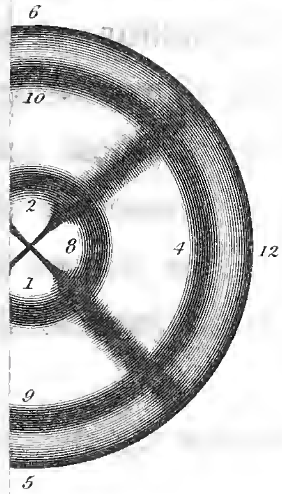

Fig. 3.

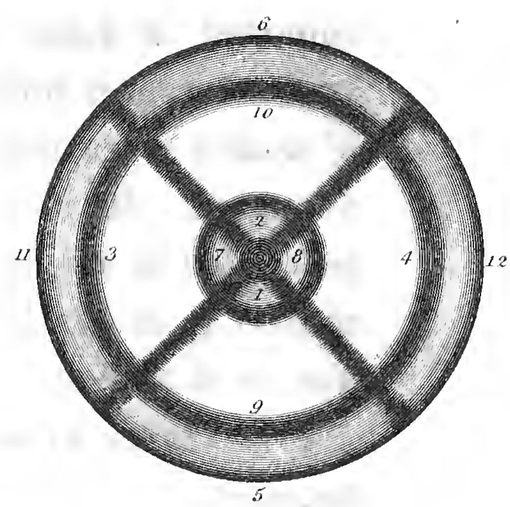

Fig. 4.

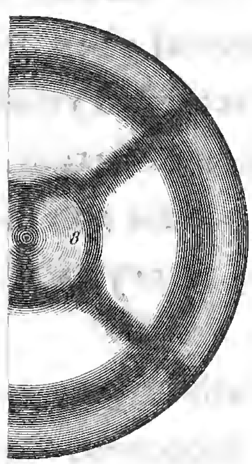

Fig. 2.

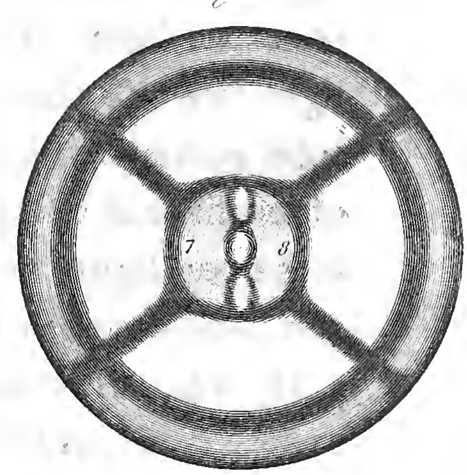

Fig. 5.

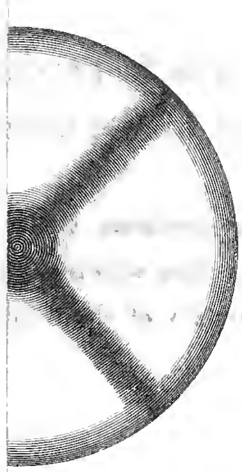

Fig. 6.

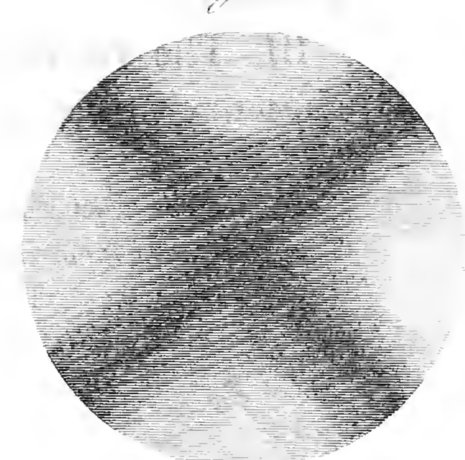

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Fig. 1.

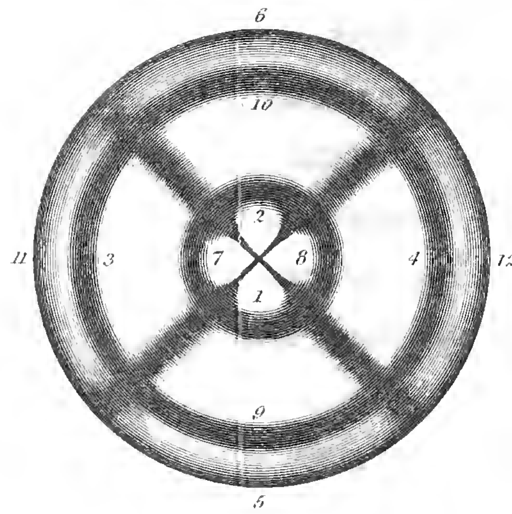


Fig. 3.

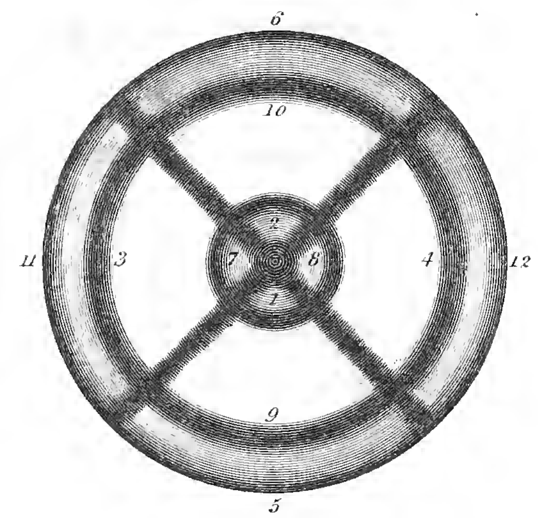


Fig. 4.

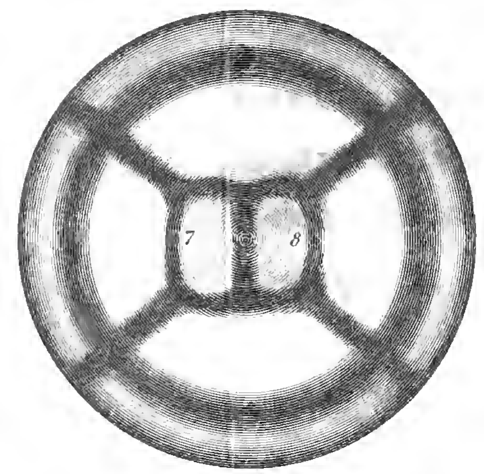


Fig. 2.

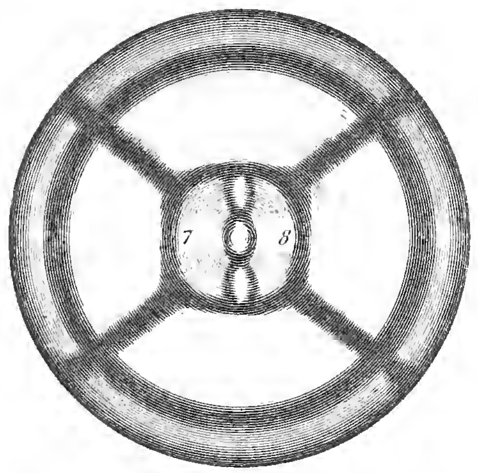


Fig. 5

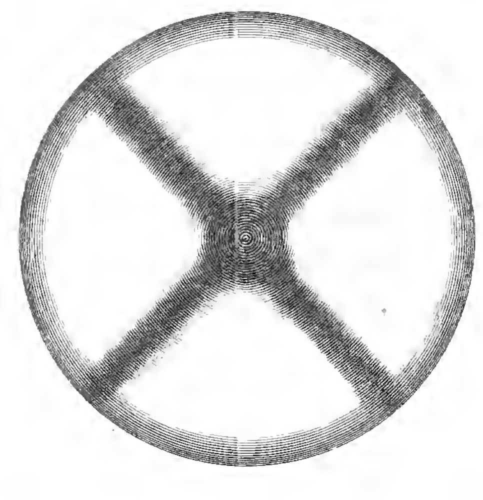
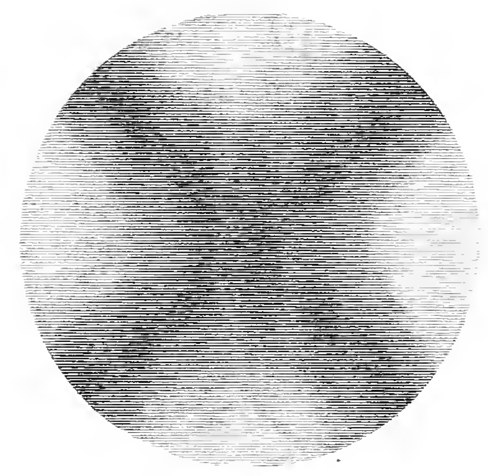


Fig. 6.



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talline, but to the diameter which forms the axis of vision. For if the variation of density were related to the centre, the sphere would have a symmetrical structure, and like the glass ball already mentioned, would exhibit the same figure in every position.

IV. It is highly probable, that this peculiar structure of the crystalline is necessary for correcting the spherical aberration.

I have the honour to be, &c.

D. BREWSTER.

To the Right Hon. Sir Joseph Banks, Bart. G. C. B. P. R. S.

Edinburgh, May 20, 1816.

XXII. *Some farther account of the fossil remains of an animal, of which a description was given to the Society in* 1814. *By* *Sir* Everard Home, *Bart. V. P. R. S.*

Read June 13, 1816.

For the materials of the former Paper I was indebted to Mr. Bullock, who has in his possession the skull, a great number of the vertebræ, many mutilated ribs, and other bones of this animal, in a fossil state.

These now brought forward are in the collections of the Rev. Mr. Buckland, of Corpus Christi College, Oxford, and of Mr. Johnson of Bristol, who have very kindly allowed me to make use of them upon this occasion. Mr. Johnson has been a collector of specimens of fossil remains for 25 years; during several summers, he devoted five or six weeks at a time to a close inspection of the cliffs and beach at Lyme. In the summer of 1814, with the assistance of a friend, at great personal risk, he dug out of the cliff, the bones of the pectoral fin; the single bone, he states, was immediately connected with the scapula, and was imbedded in marle; a representation is annexed.

From these valuable specimens I am enabled, in a great measure, to complete the account of the skeleton of this very extraordinary animal, and, what is of infinitely more consequence, to determine the class to which it belongs.

The structure of the vertebræ explained in the former paper, made it evident, that the animal in its mode of progressive motion resembled fishes; it could not, however, be determined that it was in all respects a fish, till the articulations of the ribs with the vertebræ, and the bones of which the pectoral fin is composed, had been examined.

In all animals that breathe by means of lungs, each rib, to admit of its being raised and depressed, is articulated both to the body and the transverse process of the vertebræ; but in fishes, the ribs requiring no such motion, are only connected to the bodies of the vertebræ laterally, so as not to interfere with their extensive motion on one another. In this animal, the ribs are placed in this respect like those of fishes; they are uncommonly large, and the chætodon from Sumatra, the skeleton of which is described by Mr. BELL in the first part of the 83d vol. of the Philosophical Transactions, is the only fish I know of in which the ribs bear the same proportion to its size. The form of the scapula, as well as of the bones of the pectoral fin, is entirely different from those of the whale, but bears a resemblance to that of the same parts in the shark, so that it is only necessary to compare them together as represented in the annexed plates, to recognise their similarity.

The other circumstances that confirm this skeleton being that of a fish, are the bones in a growing state having no epiphyses, as will appear from the first bone of the pectoral fin, which is represented of its natural size, having none, although when compared with the single vertebra, also represented of its natural size, the fin must have belonged to a growing animal; the ribs having been grooved longitudinally by pressure, showing the softness and toughness of their

texture, the fibrous appearance of the scapula, arising from a mode of growth only met with in the bones of fishes.

The drawings annexed to this and to the former Paper, represent the principal bones composing the skeleton of this very extraordinary animal, and they correspond sufficiently with those of fishes, to remove all doubt of its having been a fish, but different from any fishes now in existence; for although the pectoral fins bear a certain resemblance to those of the shark, there is none between many of the other parts, particularly the long projecting snout and the conical teeth.

In truth, on a due consideration of this skeleton, and of that represented in the 13th vol. of the An. Mus. p. 424, we cannot but be inclined to believe, that among the animals destroyed by the catastrophes of remote antiquity, there had been some at least that differ so intirely in their structure from any which now exist, as to make it impossible to arrange their fossil remains with any known class of animals.

EXPLANATION OF THE PLATES.

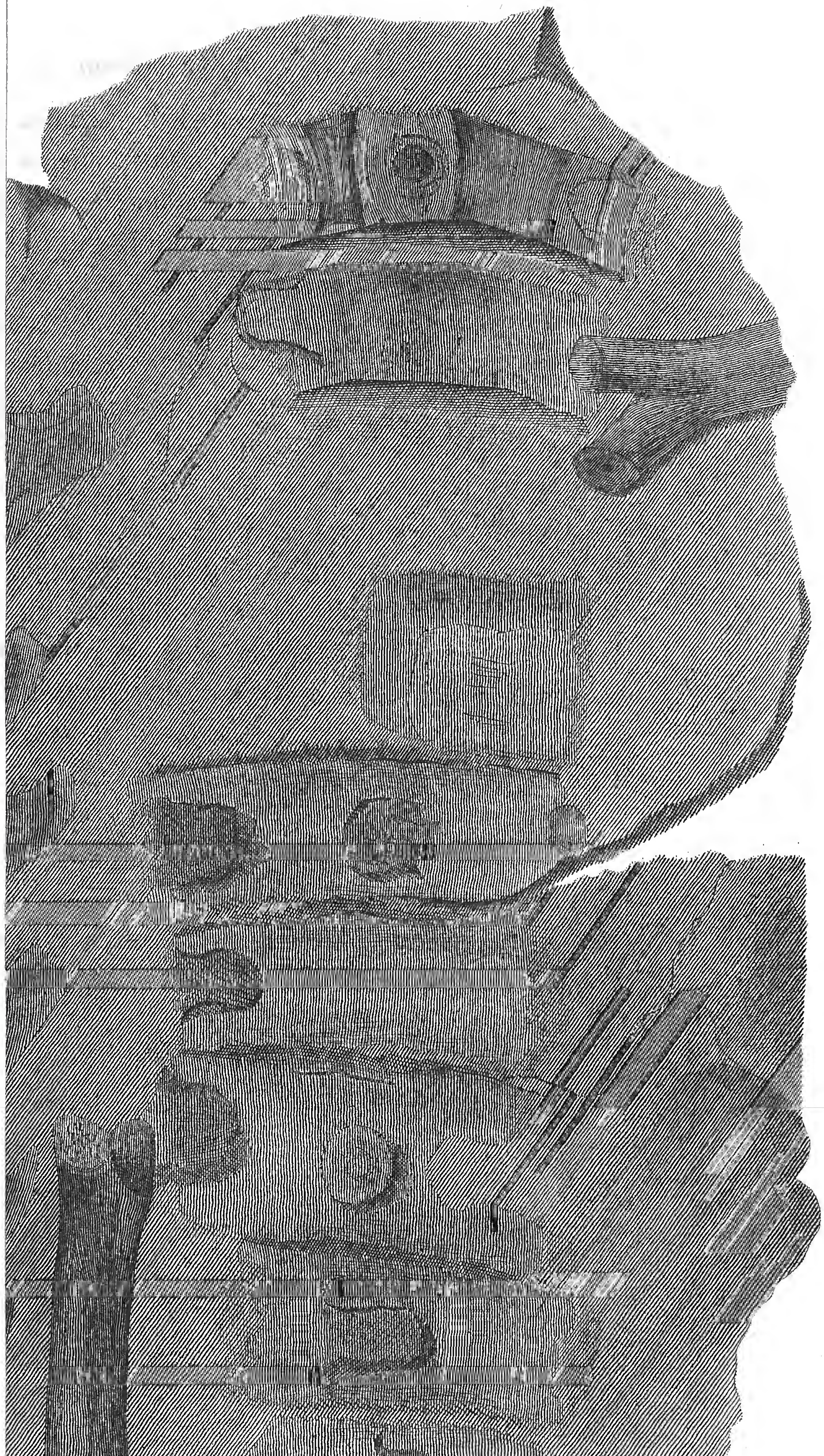
The drawings were taken from specimens of bones in very different stages of growth, but undoubtedly belonging to the same species of animal.

PLATE XIII.

Shows the manner in which the ends of the ribs correspond with the impressions on the vertebræ formed to receive them. The parts of the natural size. From a specimen of the Rev. Mr. BUCKLAND.

PLATE XIV.

A single vertebra of the natural size: it shows that the



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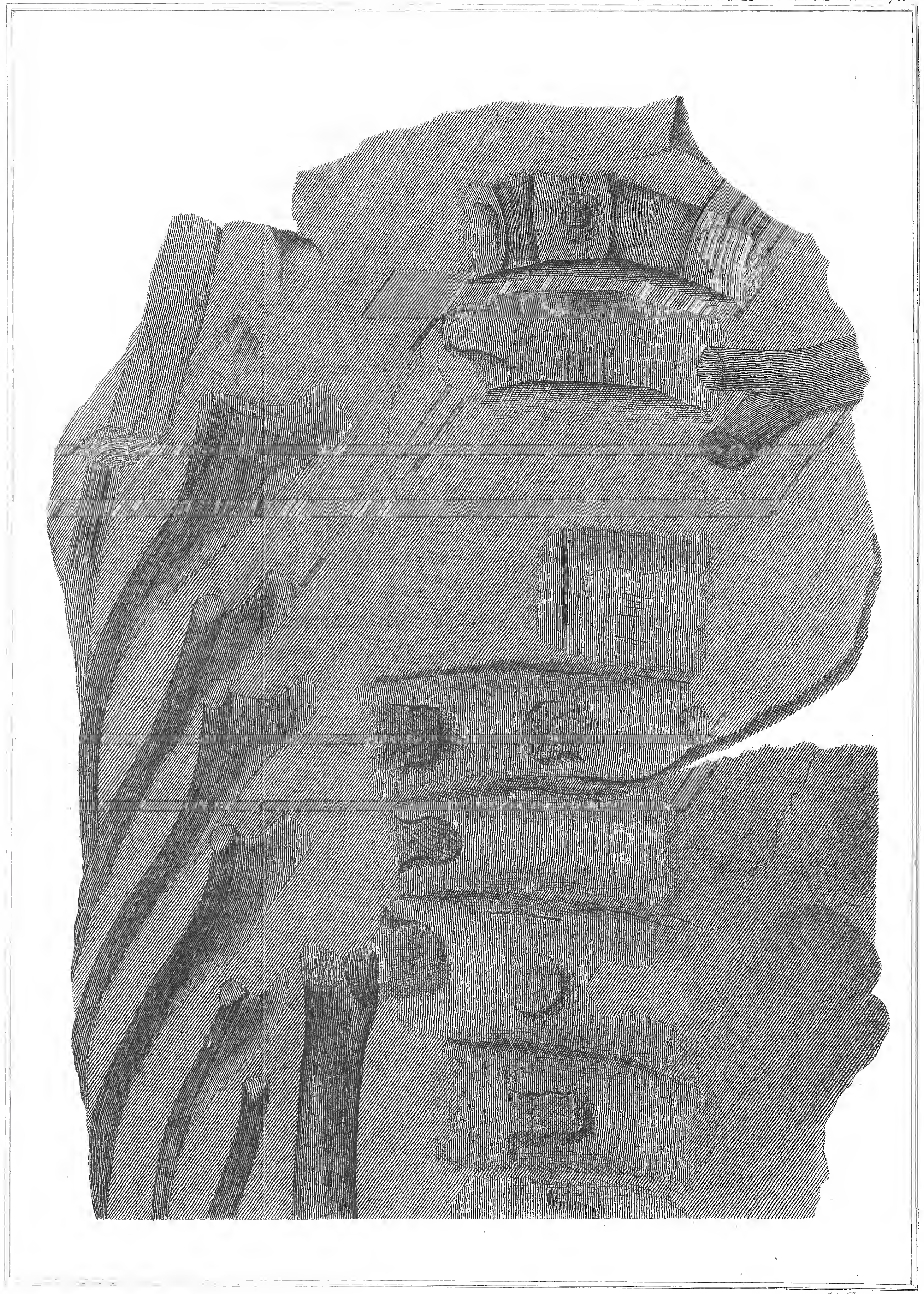

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J. Basire, sc.

Fig. 1.

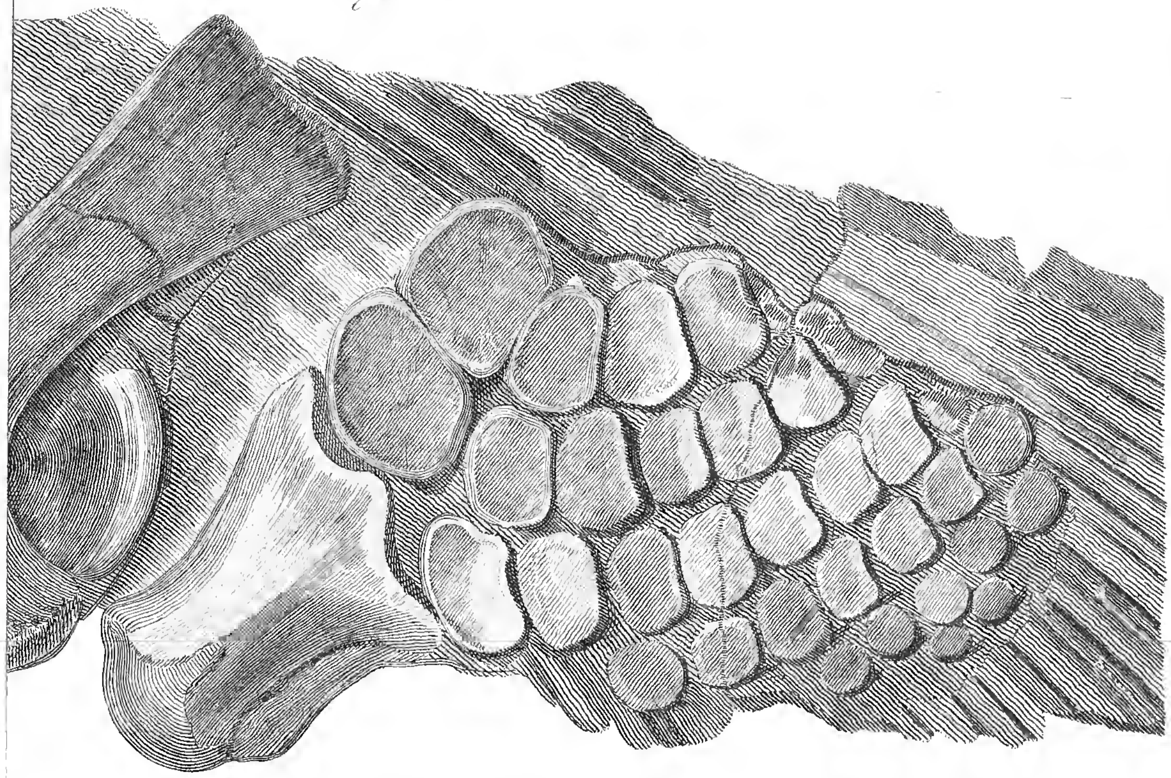
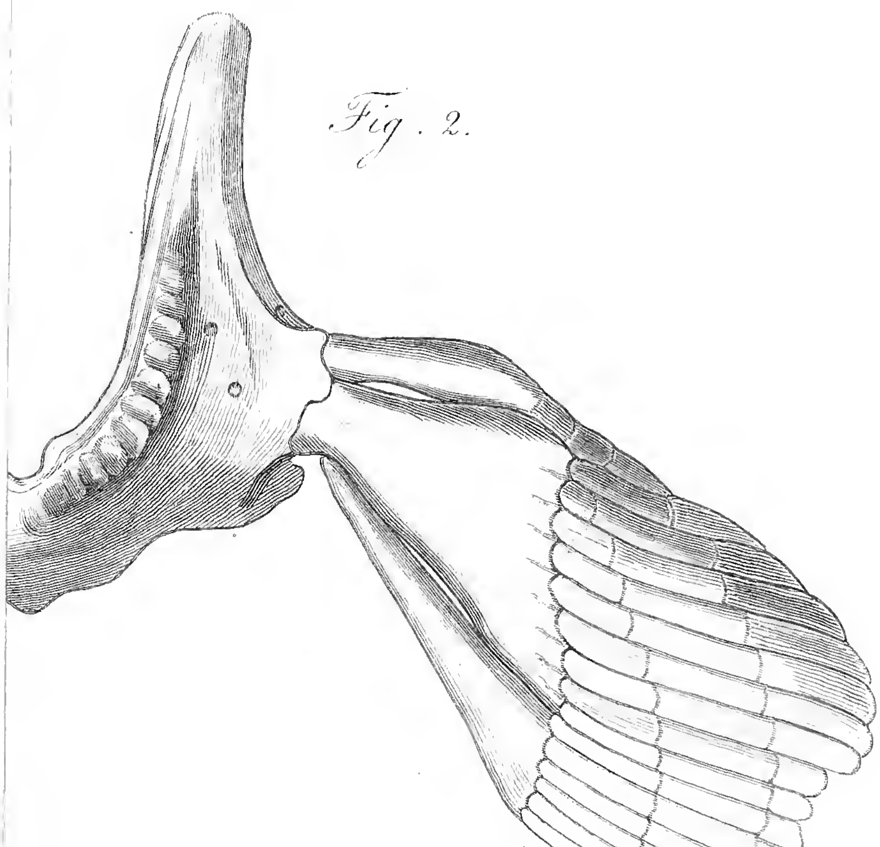


Fig. 2.



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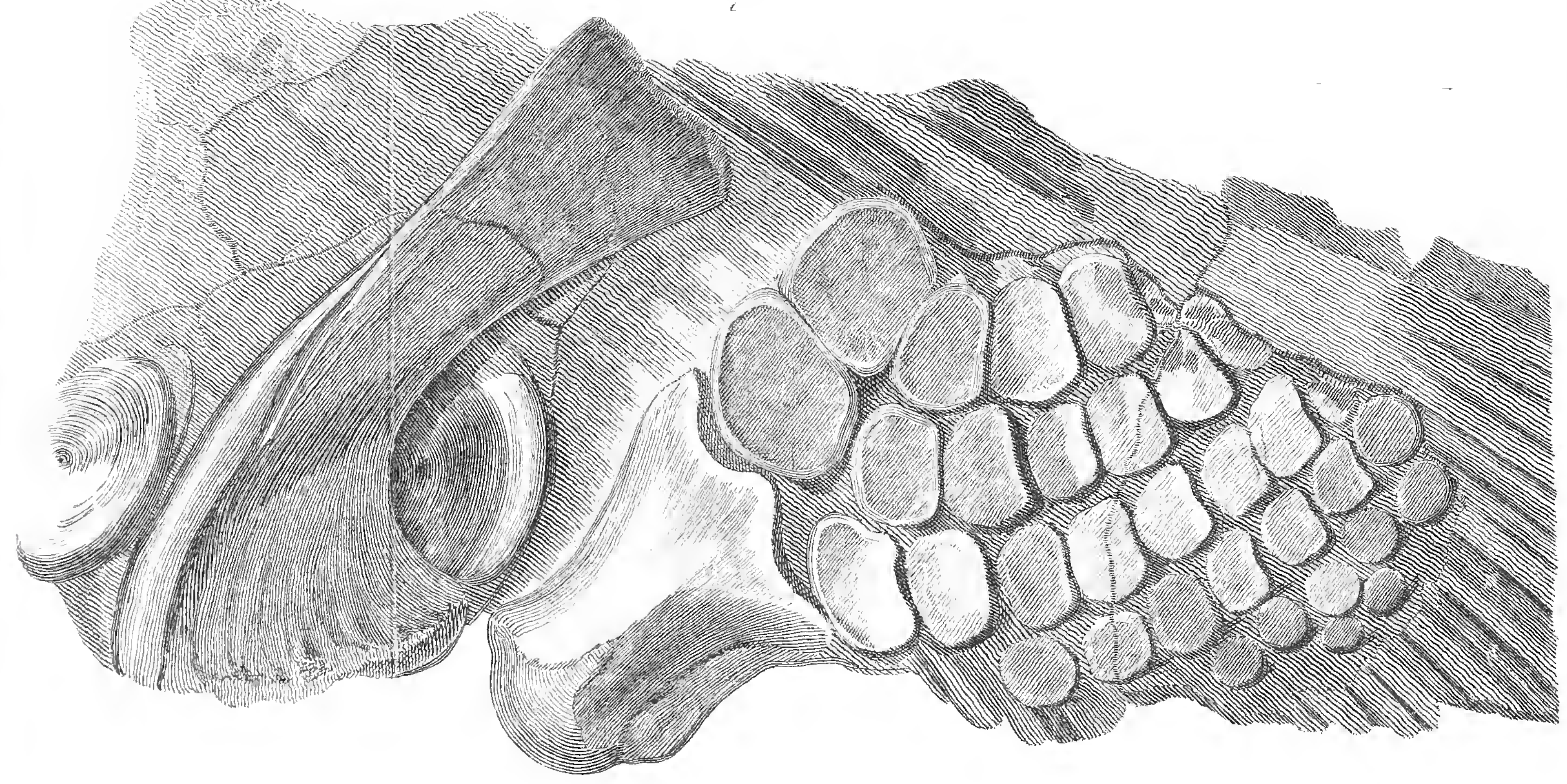
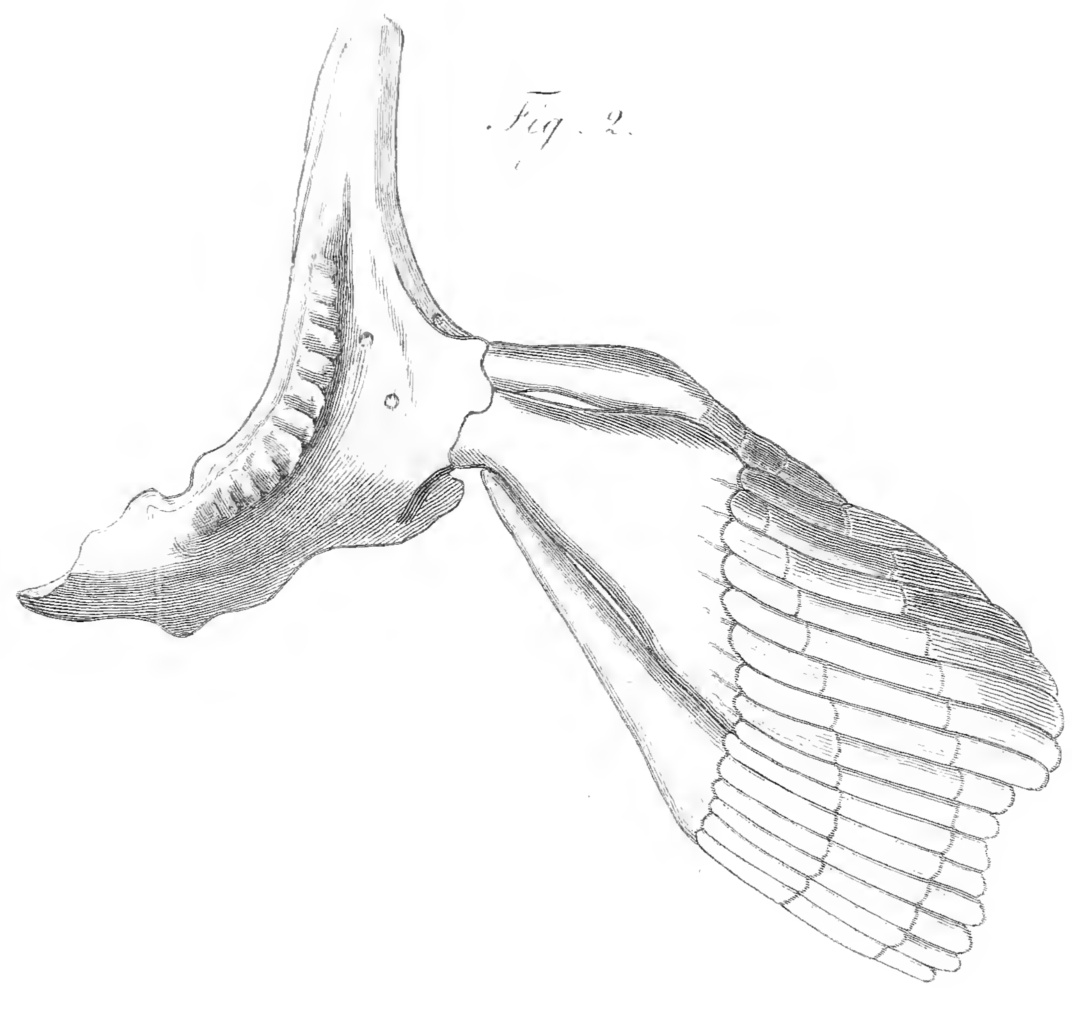


Fig. 2.



Wm. Clift del.

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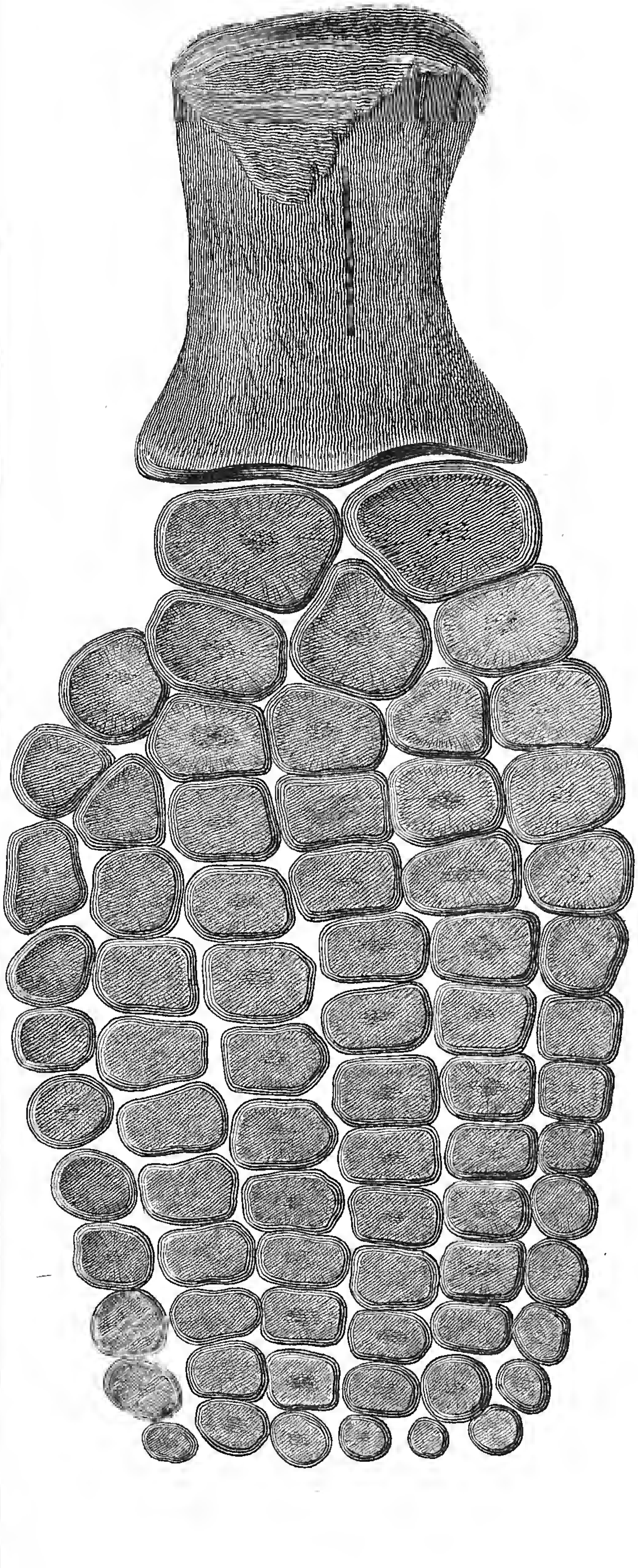

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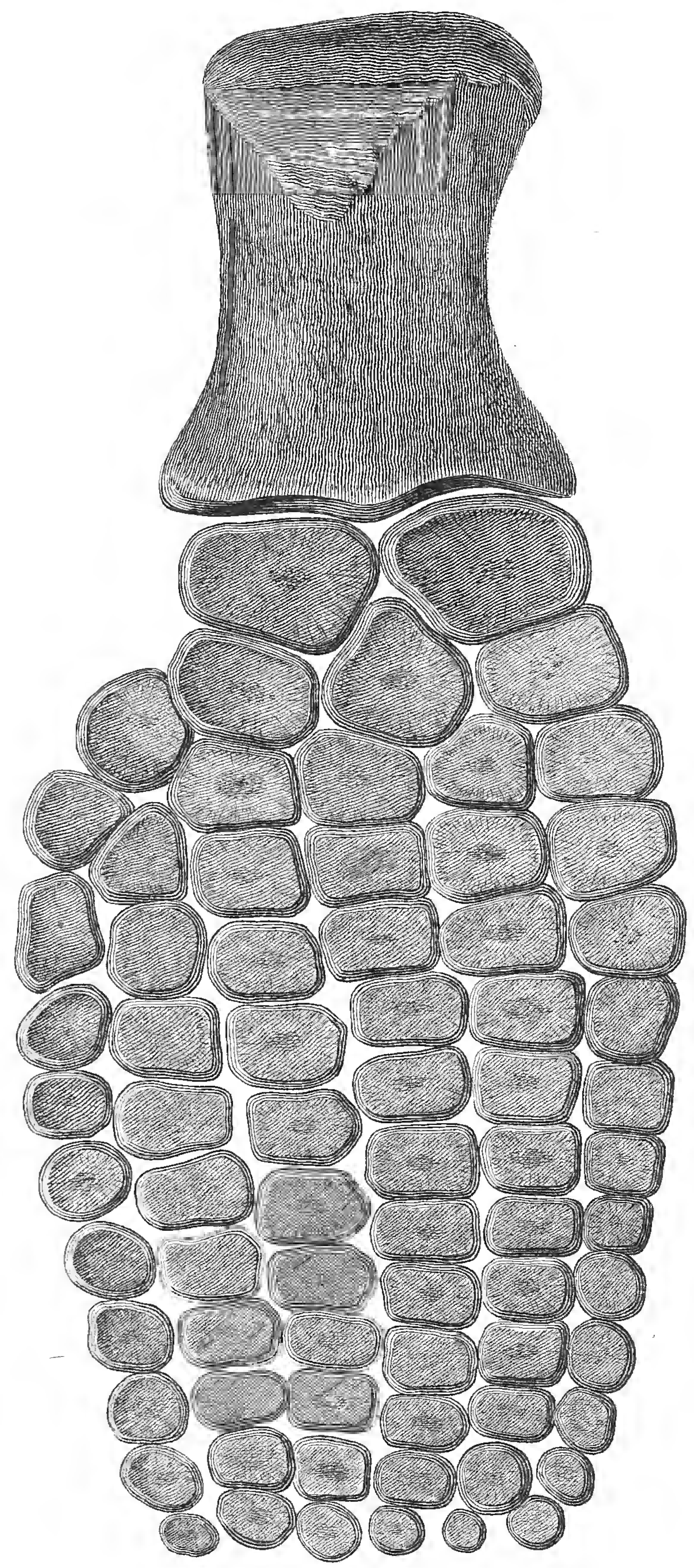


W. Clift del. Jas. Basire sc.

animal grows to a large size, and points out the particular parts of the circumference to which the rib is attached. From a specimen of the Rev. Mr. Buckland.

Plate XV.

Fig. 1. Shows the scapula in a more perfect state than in the plate belonging to the former Paper; the spine, which in the other specimen was broken off, in this is in its place, although very nearly detached; it also shows the bone intermediate between the scapula and the small bones of the fin, with some of these bones on a scale of four inches to a foot. From a specimen of the Rev. Mr. Buckland.

Fig. 2. The scapula and bones of the pectoral fin in the squalus acanthus, natural size.

Plate XVI.

The bones of the pectoral fin, natural size. From a specimen of Mr. Johnson.

XXIII. *Farther observations on the feet of animals whose progressive motion can be carried on against gravity. By Sir* Everard Home, *Bart. V. P. R. S.*

Read June 27, 1816.

Since my observations on the foot of the lacerta gecko and fly were laid before the Society, Mr. Bauer, of Kew, has made drawings of the feet of both these animals; in the hands of an artist who has attained such excellence in the correct representation of objects highly magnified, the mechanism by which the feet are fitted for supporting the weight of the animal against gravity, is much better shown.

Mr. Bauer has not confined his labours to these objects, but has applied the powers of the microscope to the examination of the feet of other insects, and has enabled me to show, that the principle on which progressive motion against gravity depends, is very extensively employed by nature in the structure of the feet of insects; indeed the means employed for this purpose are so various, as to form characters by which many genera may be distinguished.

I shall not enter farther into this enquiry than to show some of the varieties of the structure.

My friend, Dr. Leach, whose researches in entymology fit him so admirably for the purpose, will, I trust, prosecute this subject, and ascertain the peculiarities that belong to the feet of different insects, fitting them for their peculiar habits

of life; which will assist him in giving a better arrangement of the subdivisions of this numerous class than has been hitherto done.

This structure of the feet of insects, now that it is known, can be very readily demonstrated by looking at the movements of the feet of any insect upon the inside of a glass tumbler, through a common magnifying glass; the different suckers are readily seen separately to be pulled off from the surface of the glass, and reapplied to another part.

The pockets on the under surface of the toes of the lacerta gecko, as they are represented in Plate XVII, show that what looked like a pectinated edge when seen through a common magnifying glass, consists of a complex structure, composed of rows of a beautiful fringe, which are applied to the surface on which the animal walks against gravity, while the pockets themselves are pulled up by the muscles attached to them, so as to form the cavities into suckers.

In the blue bottle fly, it will be seen, in Plate XVIII, that the suckers are two in number, that they are connected to the last joint of the toe, immediately under the root of the claw, and have a narrow infundibular neck attached to the toe, which has the power of motion in every direction; when these suckers are to be applied, they are separated from each other, and the membrane of each is expanded so as to increase the surface; but when disengaged, they become nearly closed, and are brought together, so as to be confined within the space between the two claws.

The external edge of each sucker is beautifully serrated, and the concave surface is granulated. When the fly is walking against gravity, and its motions are observed, they all appear

to be the result of muscular action, regulated by the will of the animal.

All the six toes are supplied with suckers.

In the horse fly, the tabanus of FABRICIUS, the suckers differ from those of the blue bottle fly in being three in number, in all other respects they are the same. In this fly, when the suckers are not used, the two outer ones close in before the other, and are only spread out when they are to be brought into use.

In the yellow Saw fly, the cimbex lutea of FABRICIUS, the suckers are differently situated from those of the fly; they are placed upon the under surface of the four first joints of the toes, one sucker upon each. These suckers are spoon shaped; they are represented in Plate XIX. The exterior part is thin and pellucid, but at half their depth they suddenly become thicker in their coats, forming a ridge at this part which gives the appearance of an inner cup, but this is a deception; the exterior membranous portion is alone expanded on the surface to which the sucker is applied, and the neck of the sucker forms the vacuum.

All the six feet have suckers.

The apparatus which has been described to support the animal in its progressive motion, is also applied to other purposes. In the great water beetle, the dytiscus marginalis, in which there is no appearance of suckers on the under side of the feet of the female, they are placed on the three first joints of the first and second pair of feet of the male, as is shown in Plate XX.; from which it is evident, that such suckers are used to retain the female in the embrace of the male. In the male, the three first joints of the feet of the

fore legs have the form of a shield, the under surface of which is covered with suckers, one very large, a second one-third smaller, and all the rest very small. In the second pair of feet, the corresponding joints are much narrower in proportion, and are covered on their under surface with very small suckers.

All these suckers, as is seen in Fig. 13, 14, and 15, have long tubular necks, which show more plainly than in the others the mode in which the vacuum is produced; it is exactly similar in its effect to that of a piece of leather with a string in the centre, applied in a moistened state to the surface of a stone.

Having explained this apparatus, so beautifully contrived to attach the feet to the surface on which the animal moves, I shall conclude this Paper by an account of a structure of a very different kind, for the purpose of taking off the jar when the body of the insect is suddenly brought from a state of motion to a state of rest; this is met with in the grylli and locustæ. Some of them have suckers at the ends of the toes, others have not.

In a species of gryllus with a corcelated thorax, brought from Abyssinia, by Mr. Salt, the feet are made up of three joints; on the under surface of the first are three pair of globular cushions, filled with an elastic fibrous substance, looser in its texture towards the circumference, which renders it still more elastic; under the second joint is one pair of similar cushions, and under the last joint, immediately between the claws, is a large oval sucker. A similar sucker is met with between the claws in a British grasshopper, the acrydium biguttulum (Latr.) These are common to all the six feet. They are represented in Plate XXI.

In the locusta varia, whose feet have four joints, under the first are two very small globular cushions, and two large oval ones; under the second, a corresponding pair of oval ones; and under the third, a pair of cushions different from the others, in being much larger, globular, and semi-transparent; there is no sucker between the claws, and this insect has no power of supporting itself against gravity.

As the flea has powers of jumping not exceeded by any other insect, it was natural to expect a similar apparatus under its feet; but as no such cushions are met with, we must conclude that the lightness of its body rendered them unnecessary.

EXPLANATION OF THE PLATES.

Plate XVII.

Represents six different views of the third toe of the fore foot of the lacerta gecko.

Fig. 1. Is the upper surface of the toe, to show the manner in which it spreads laterally.

Fig. 2. The under surface of the same toe, to show the orifices of the pockets or suckers. These two figures are magnified 100 times.

Fig. 3. Two portions of two contiguous suckers, showing that the fringed termination is only continued from the ends of the alternate membranous partitions. The parts are magnified 2500 times.

Fig. 4. A front view of a longitudinal section, to show the bones and muscles.

Fig. 5. A side view of a longitudinal section. These two are magnified 100 times.

Fig. 6. A side view of a portion of some of the suckers, showing the insertion of the muscles, magnified 2500 times.

Plate XVIII.

Fig. 1. A left front leg of the blue bottle fly, musca vomitoria, magnified 100 times.

Fig. 2. A view of the under side of the last joint of the toe, with the two suckers expanded, as seen when the fly is walking against gravity.

Fig. 3. Side view of ditto.

Fig. 4. Upper side of ditto. These three figures are magnified 6400 times.

Fig. 5. View of the under side of a single sucker of a dead fly.

Fig. 6. Side view of ditto.

Fig. 7. Upper side of ditto. These three figures are magnified 6400 times.

Fig. 8. Left front leg of bibio febrilis (Latr.) Magnified 100 times.

Fig. 9. The under side of the last joint of the toe of ditto.

Fig. 10. Side view of ditto.

Fig. 11. Upper side of ditto. These three figures are magnified 6400 times.

Plate XIX.

Fig. 1. The left front leg of cimbex lutea, (Fabr.) *a*, thigh; *b*, shank; *c*, calces; *d*, toe; *e*, plantar suckers; *f*, sucker between the claws; *g*, unguis or claw.

Fig. 2. Side view of ditto.

Fig. 3. Under side of the left front leg. These three magnified 100 times.

Fig. 4. Last joint of toe, upper side, magnified 400 times.

Fig. 5. Under side of toe, and extremity of the shank, magnified 400 times.

Fig. 6 and 7. Upper and side views of two joints of the toe, magnified 400 times, to show the plantar suckers.

Fig. 8 and 9. Calces, magnified 1600 times, to show the form of the sucker with which each is terminated.

Plate XX.

Fig. 1. The left front leg of the male dytiscus marginalis.

Fig. 2. Ditto, under side. They are both magnified 25 times.

Fig. 3. Tarsus of ditto.

Fig. 4. Side view of ditto.

Fig. 5. Under side of ditto. These three are magnified 100 times.

Fig. 6. Front view of the sucker (*a*) fig. 5, magnified 100 times.

Fig. 7. Side view of the sucker (*b*) fig. 5, magnified 100 times.

Fig. 8. Front view of the suckers (*c*) fig. 5, magnified 400 times.

Fig. 9. Several of the suckers (*c*) fig. 5, magnified 400 times.

Fig. 10. Second or middle left leg of dytiscus marginalis, magnified 25 times.

Fig. 11. Tarsus of ditto, under side, magnified 100 times.

Fig. 12. Several of the suckers (*a*) fig. 11, magnified 900 times.

Fig. 13, 14, and 15. Suckers (*a*) fig. 11. magnified 40,000 times, to show the articulation between the peduncle and the

sucker, and the joint by which the peduncle is attached to the tarsus.

Fig. 16. The hinder left leg of dytiscus marginalis, male, magnified 25 times.

Fig. 17. The left front leg of dytiscus marginalis, female, magnified 25 times.

Plate. XXI.

Fig. 1. The left front leg of a species of the genus gryllus, (Fabr.) with a corcelated thorax, from Abyssinia, magnified 9 times.

Fig. 2. A toe of ditto, to show the under side, on which are cushions attached to the first and second joints; *a*, the oval sucker between the claws; *bbbb*, the cushions.

Fig. 3. Ditto, side view.

Fig. 4. Ditto, upper side.

Fig. 5. Vertical section of the organs, fig. 2. *b*.

Fig. 6. Longitudinal section of ditto. All these are magnified 100 times.

Fig. 7. A front view of the left fore foot of a British species of grasshopper, acrydium biguttulum, (Latr.) to show that it has the same oval sucker between the claws, and the cushions, as in the grasshopper from Abyssinia. The parts were drawn from the animal while alive. The Abyssinian grasshopper had been preserved in spirit.

Fig. 8. A side view.

Fig. 9. A back view. These three views are magnified 2500 times.

Fig. 10. A left front leg of a British species of grasshopper, locusta varia, (Fabr.) magnified 36 times.

Fig. 11. View of the underside of the toe.

Fig. 12. Side view of ditto.

Fig. 13. Upper side of the toe. These are magnified 625 times.

The cushions under the joints of the toe in this grasshopper, resemble in structure those of the Abyssinian gryllus, but differ in their form and situation. There is no sucker between the claws.

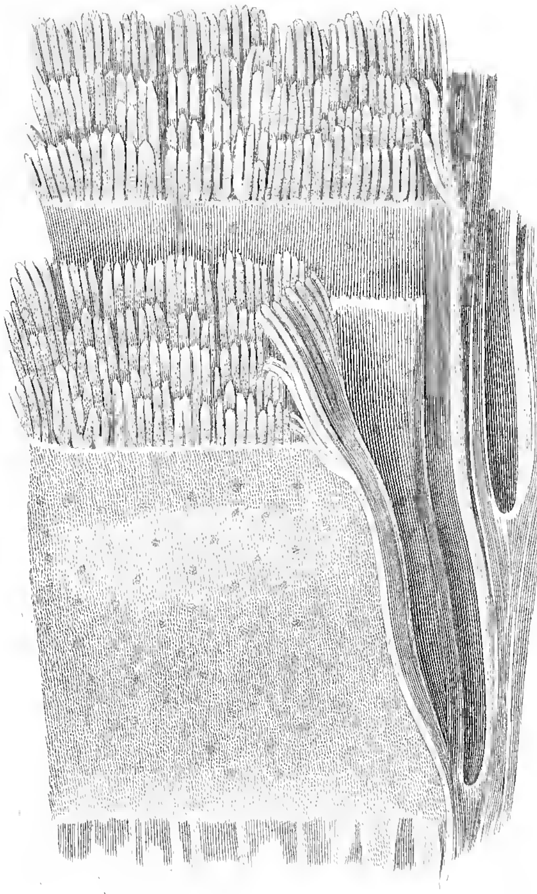
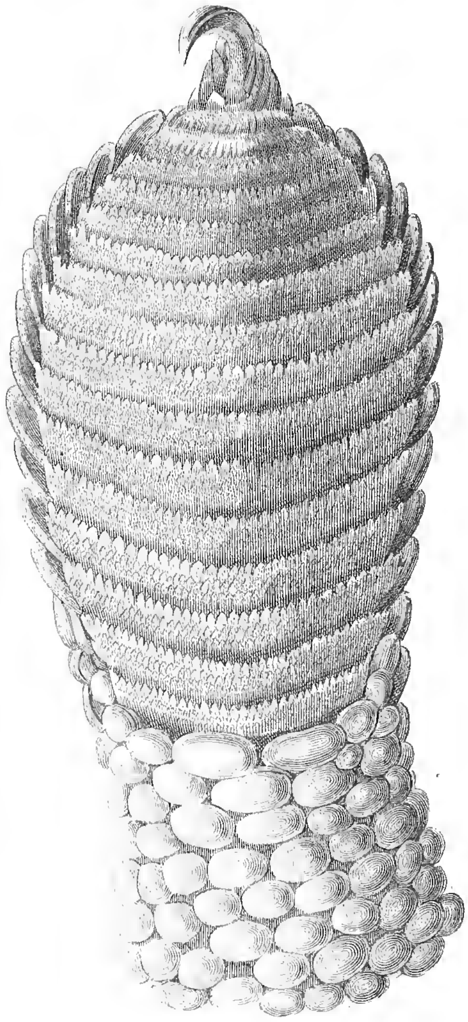


Fig. 3.



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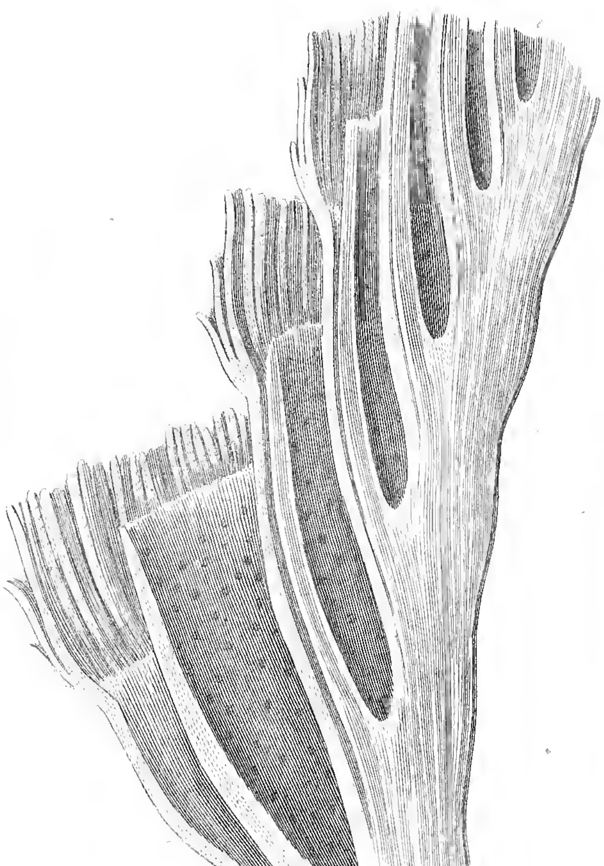

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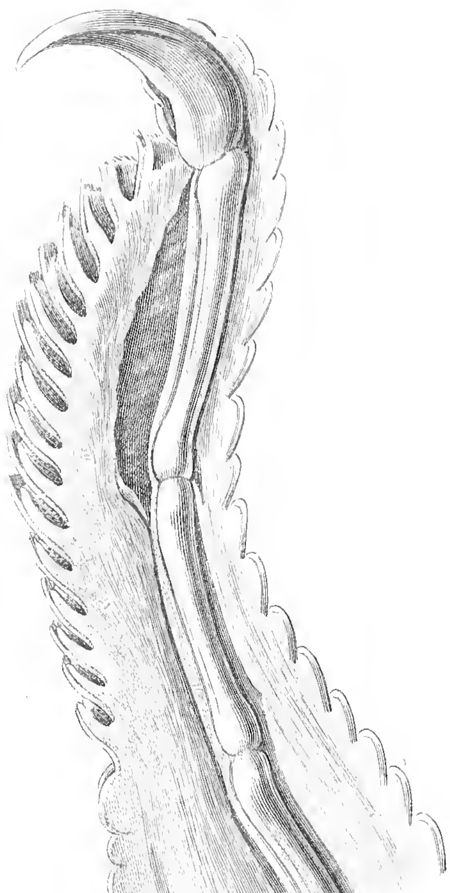

Fig. 5.

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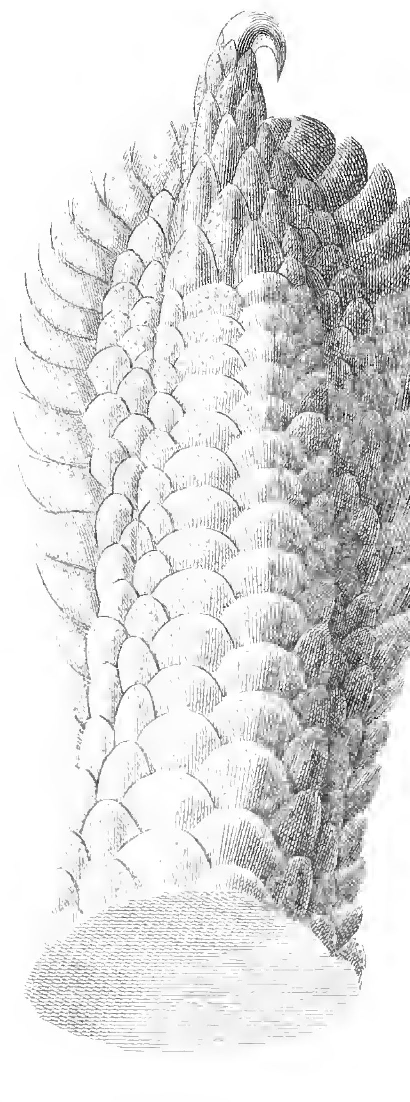

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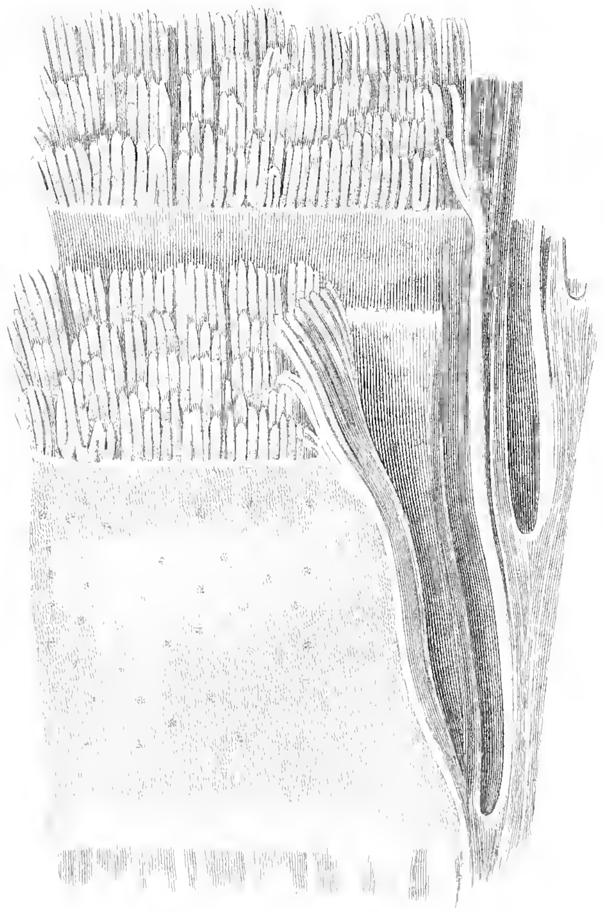

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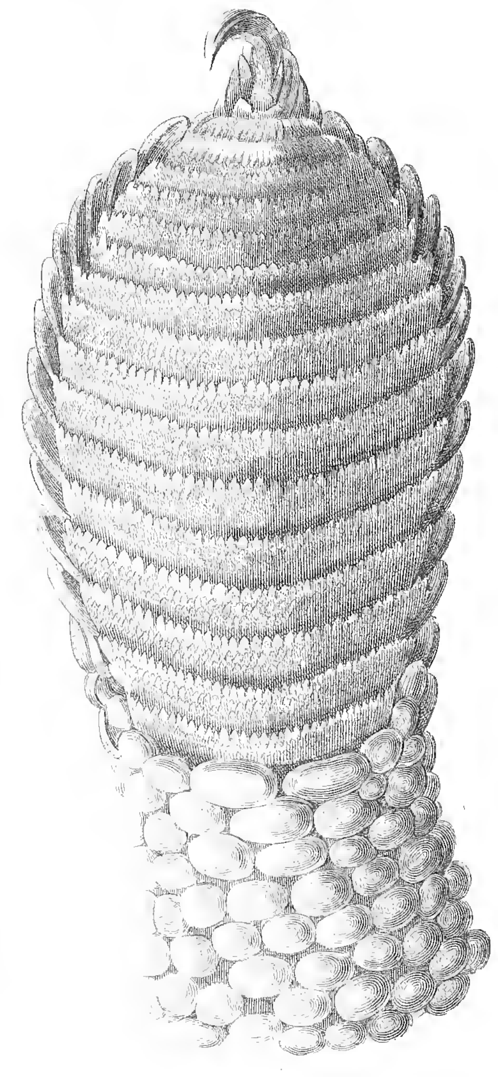

Fig. 2.

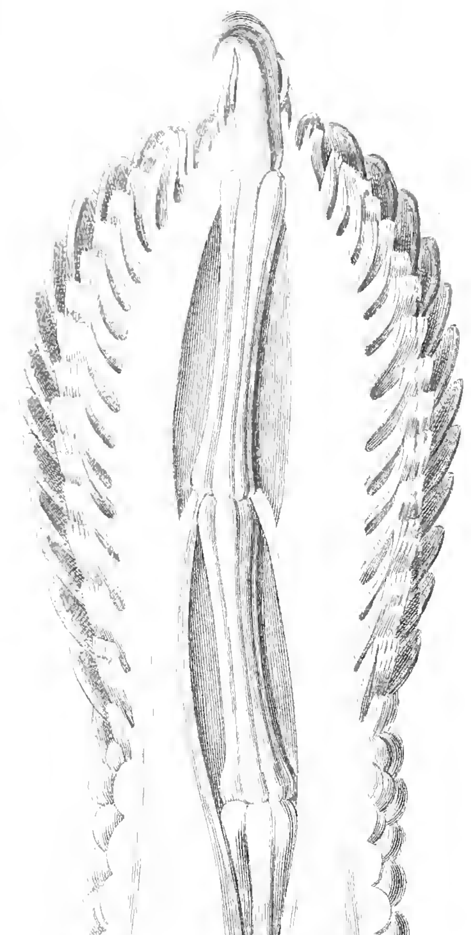

Fig. 4.

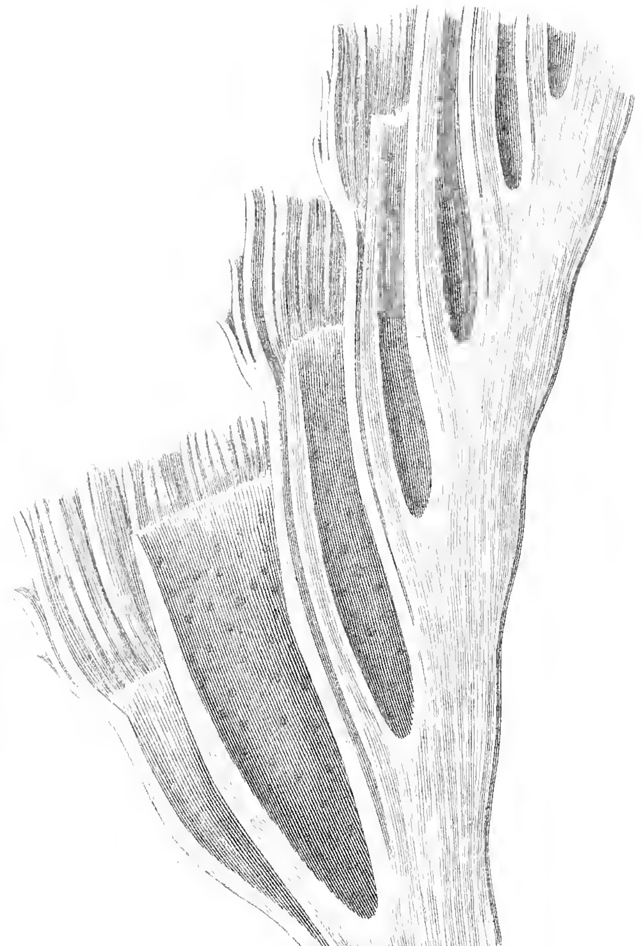

Fig. 6.

Fig. 5.

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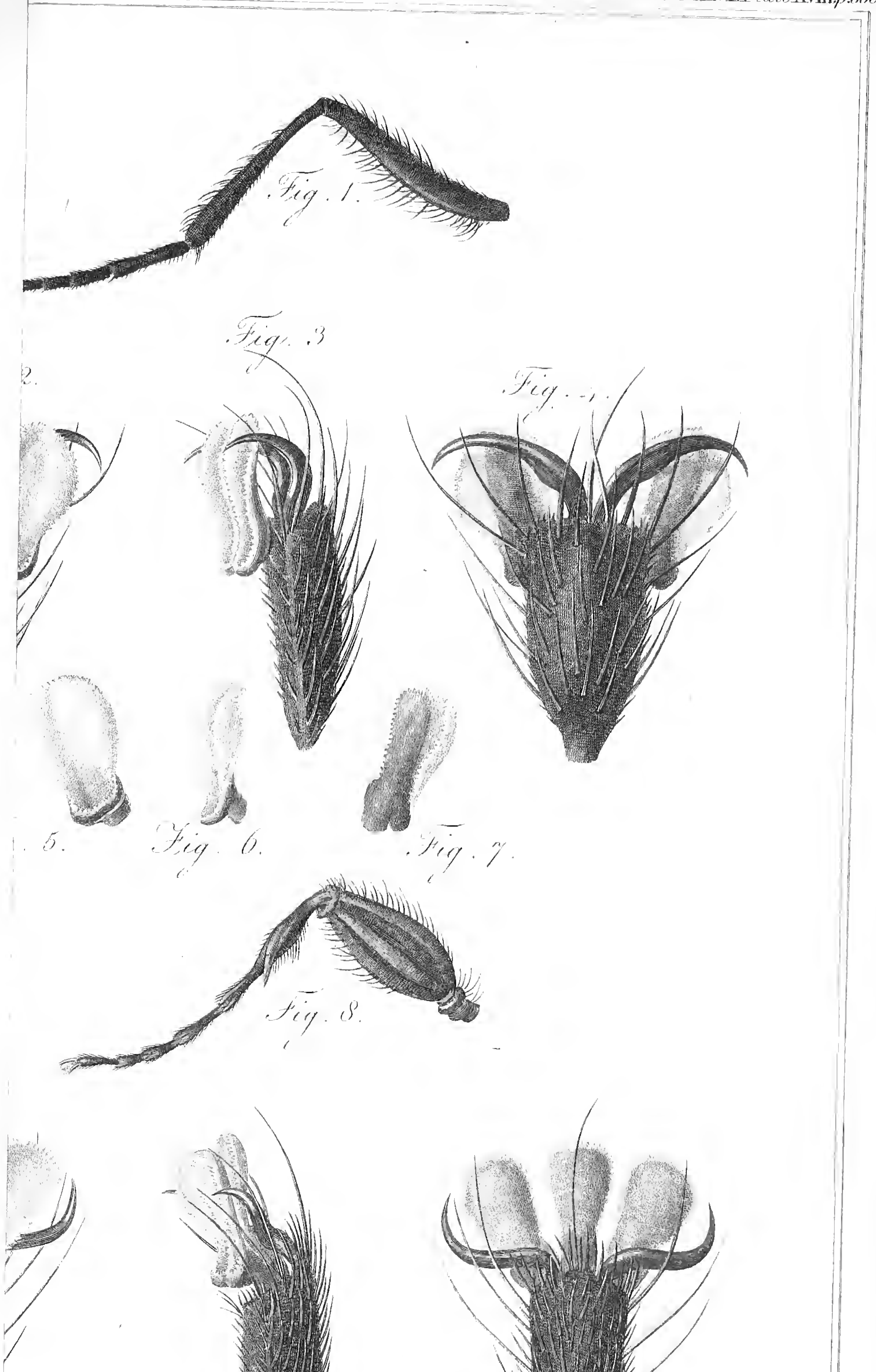

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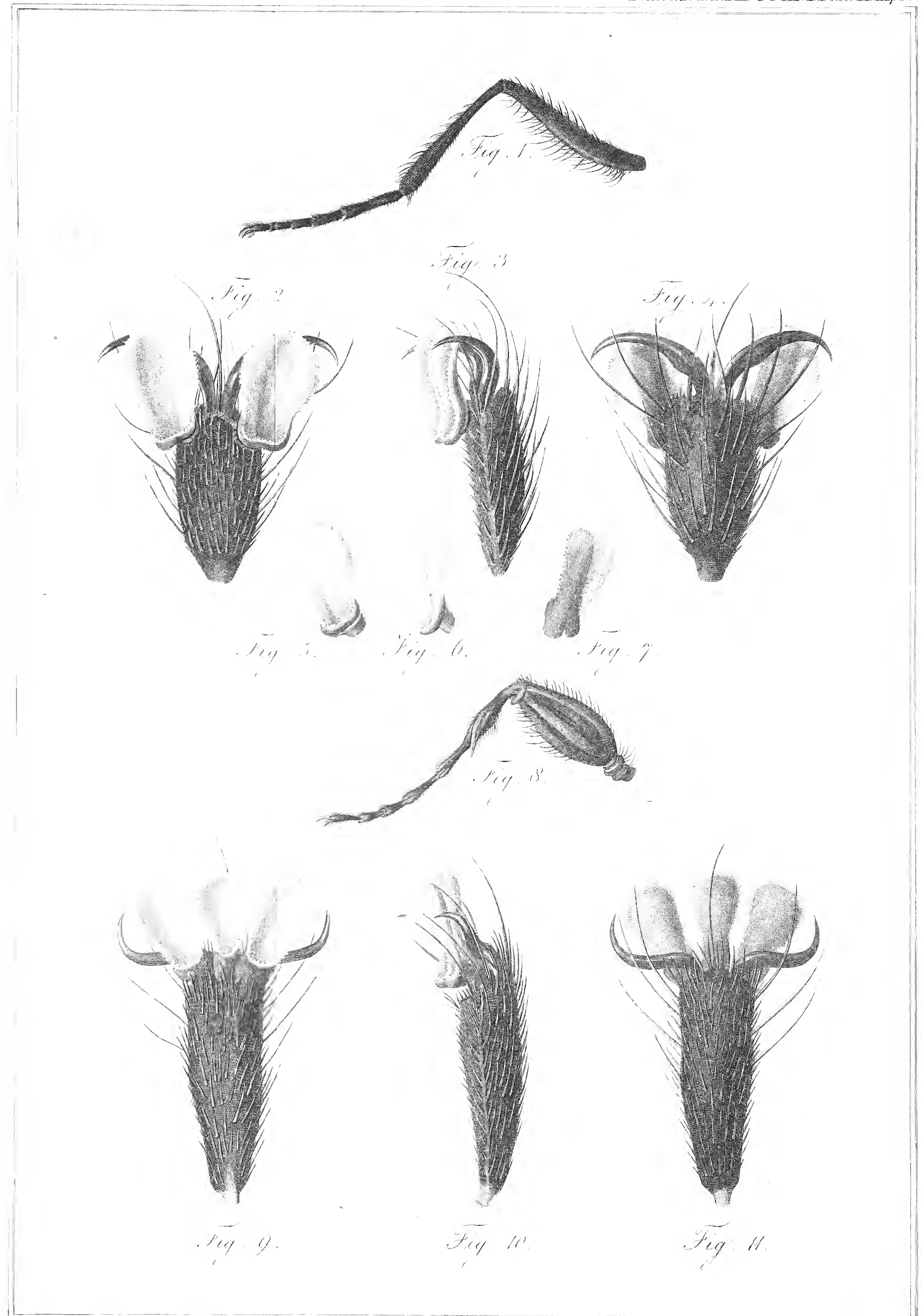

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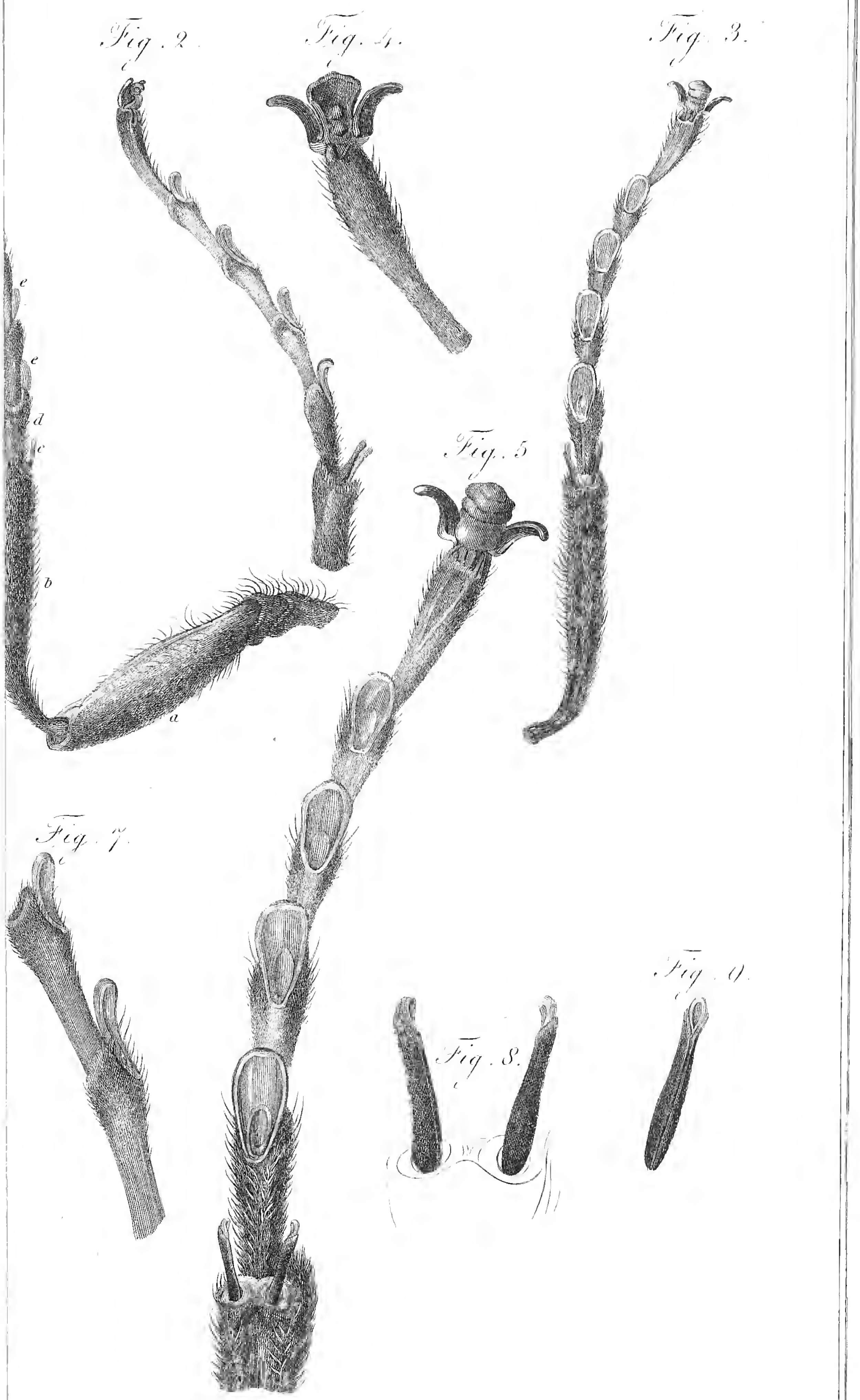

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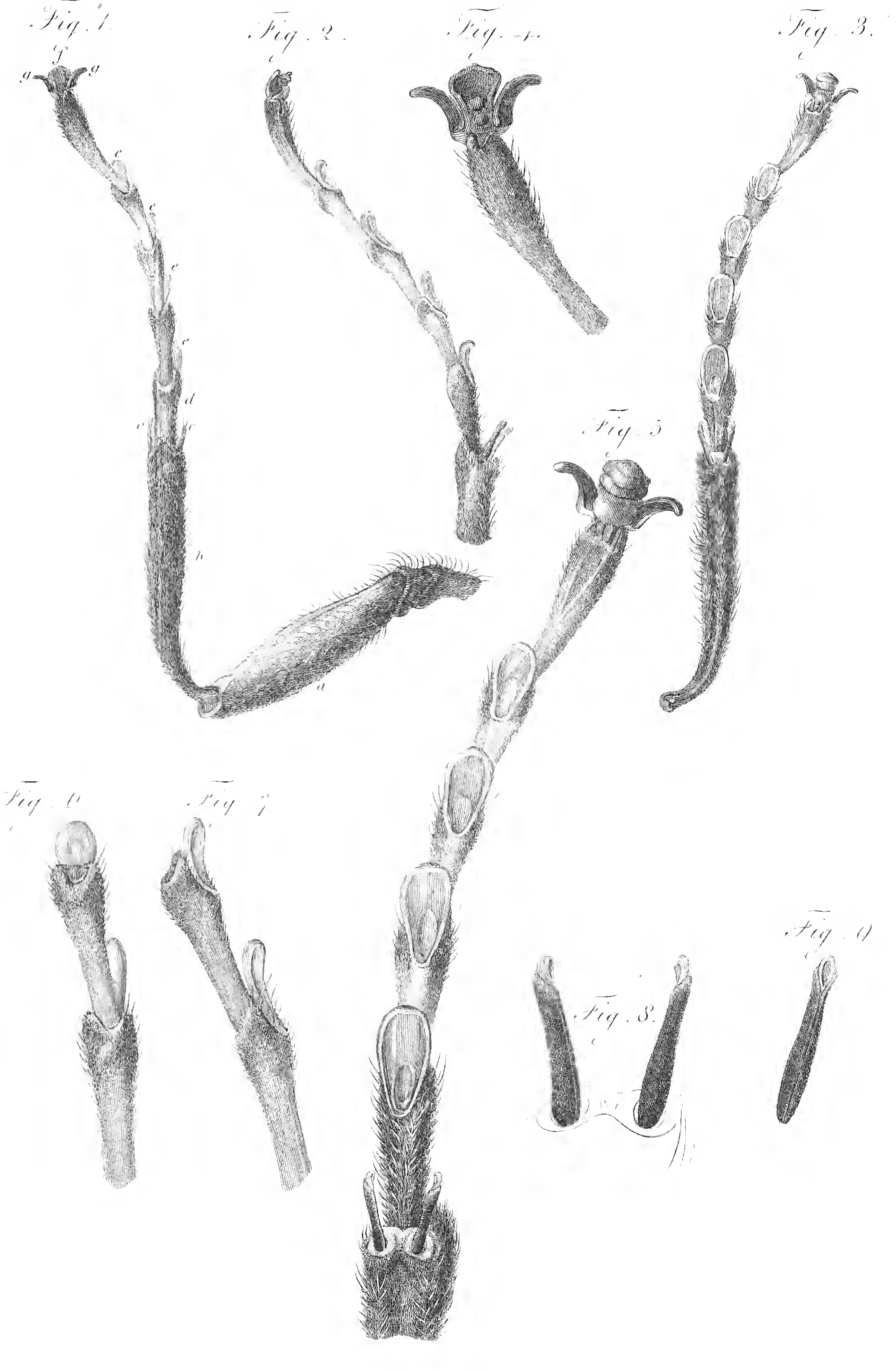

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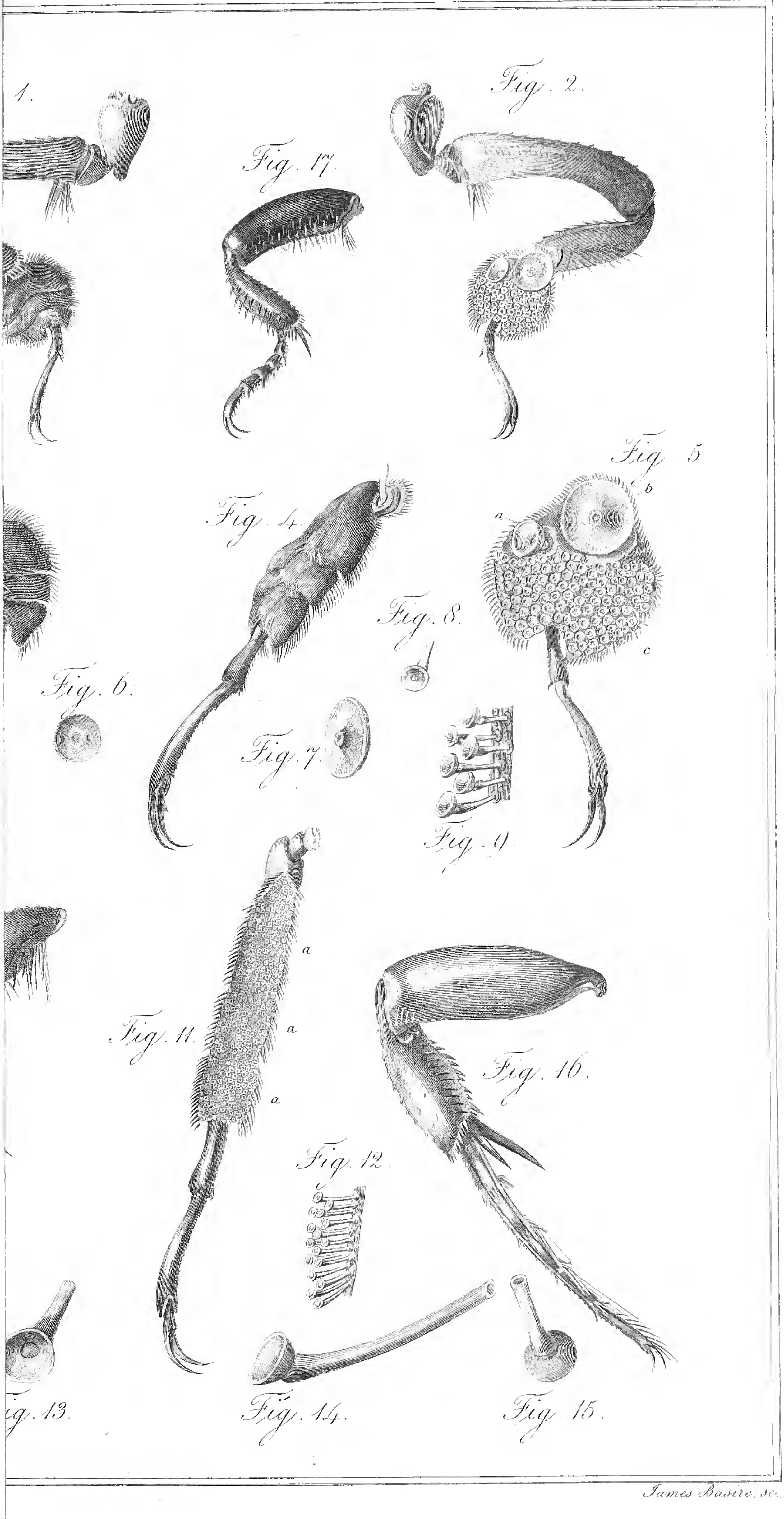

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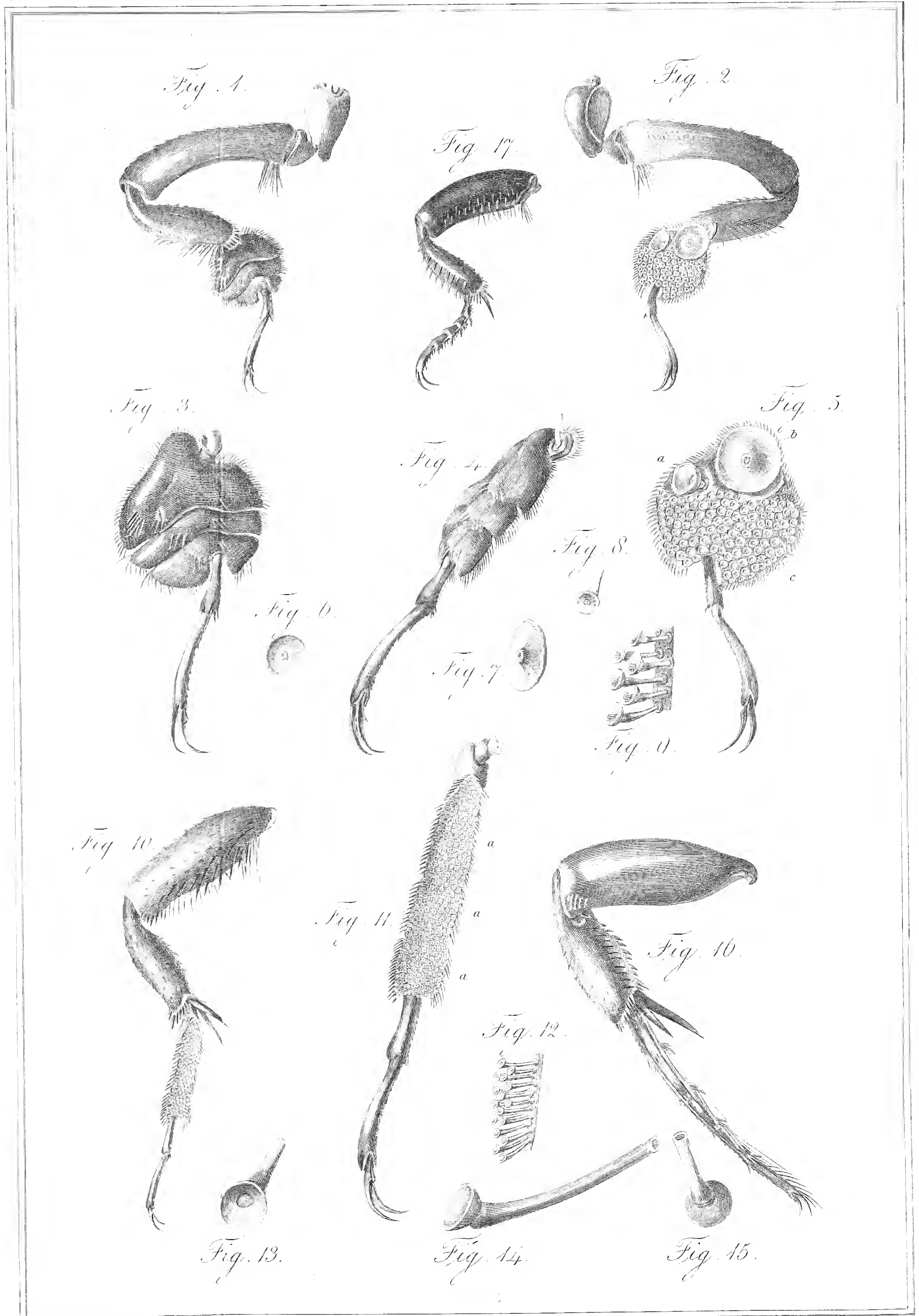

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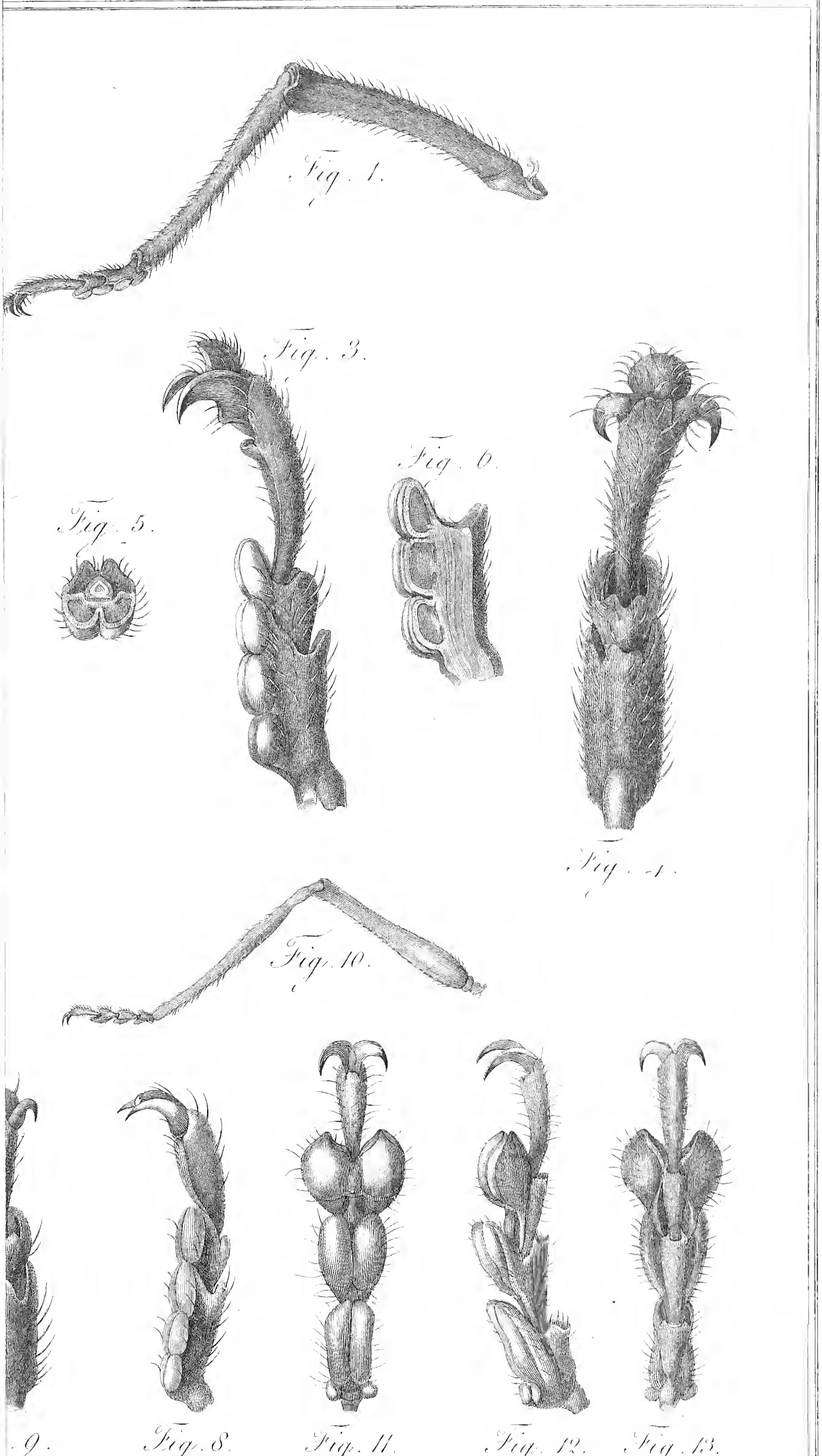

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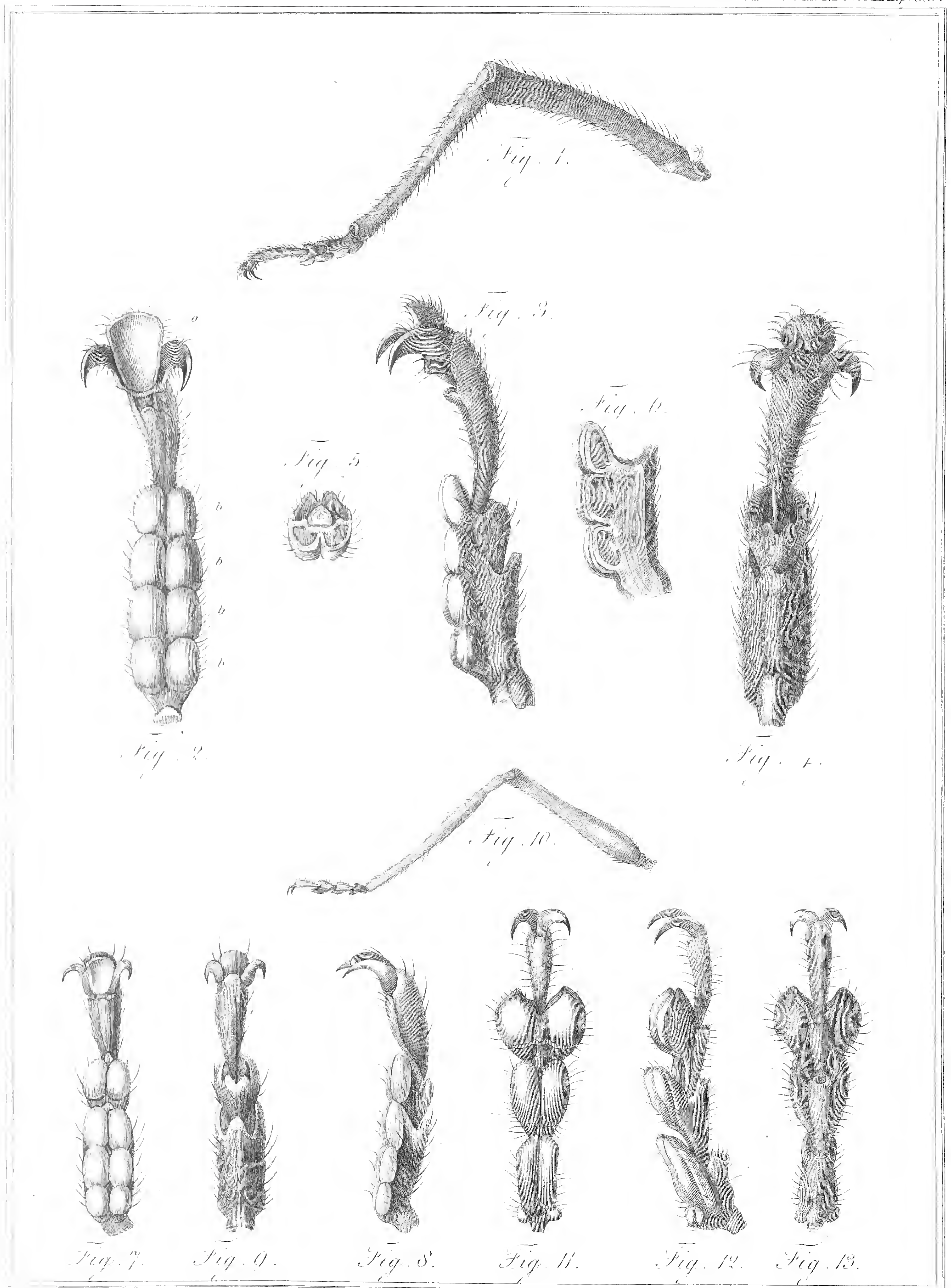

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XXIV. *A new demonstration of the binomial theorem. By* Thomas Knight, *Esq. Communicated by* W. H. Wollaston, *M. D. Sec. R. S.*

Read July 4, 1816.

It is somewhat remarkable that, amongst the various and far-fetched methods and artifices by which the binomial theorem has been obtained, no one should once have thought of the only course which seems obvious and natural. The equation $(a+x)^m \times (a+y)^m = \left\{(a+x)\,(a+y)\right\}^m$ expresses the general property of powers, whether m be positive or negative, whole or fractional; and from this equation, without the help of any artifice, the series in question is deduced.

Some investigations have been found fault with, as drawn from principles allied to the method of fluxions; whilst, on the other hand, a demonstration, taken from the "*Théorie des Fonctions*," has been represented as perfect: but I cannot help thinking that it is as much connected with the fluxional calculus as any of the rest; for it seems to make no difference whether, in $(a+x)^m$, we substitute $x+u$ for x, and take the coefficient of u, or substitute $x+\dot{x}$, and take the coefficient of $\dot{x}$. The former substitution was made because it was *known* to be equivalent to the other, and has so little apparent connection with the subject, that a student would hardly understand why it was made. The demonstration of Mr. La Croix in the Introduction to his "*Calcul Différentiel*" is

liable of course to the same objection. If we multiply $a+x$, continually, first by itself, and then by the powers successively arising, we easily see that the second term of each succeeding product is of the form $na^{n-1}x$, n being the exponent of the power: this does not require a more formal proof, and I assume it in what follows. Nor is it more difficult to perceive that, generally, m being positive or negative, whole or fractional, the following form may be assumed,

$$(a+x)^m = a^m + {}'\mathrm{A}a^{m-1}x + {}''\mathrm{A}a^{m-2}x^2 + {}'''\mathrm{A}a^{m-3}x^3 +$$

where $'\mathrm{A}$, $''\mathrm{A}$, $'''\mathrm{A}$, &c. are expressions depending on m alone, consequently, $(a+y)^m = a^m + {}'\mathrm{A}a^{m-1}y + {}''\mathrm{A}a^{m-2}y^2 + {}'''\mathrm{A}a^{m-3}y^3 +$; and because $\left\{(a+x)(a+y)\right\}^m = (a^2 + ax + ay + xy)^m =$ $= a^m\left(a+x+y+\frac{xy}{a}\right)^m = a^m(a+x+\pi y)^m$, by making $\left(\pi = 1+\frac{x}{a}\right)$ we have also

$$\left\{(a+x)(a+y)\right\}^m = a^m\left\{a^m + {}'\mathrm{A}a^{m-1}(x+\pi y) + {}''\mathrm{A}a^{m-2}(x+\pi y)^2 + {}'''\mathrm{A}a^{m-3}(x+\pi y)^3 + \right\}$$

or neglecting all the powers of y but the first,

$$= a^m\left\{a^m + {}'\mathrm{A}a^{m-1}x + {}''\mathrm{A}a^{m-2}x^2 + {}'''\mathrm{A}a^{m-3}x^3 + \right\}$$
$$+ a^m y\left\{{}'\mathrm{A}a^{m-1}\pi + 2\,{}''\mathrm{A}a^{m-2}\pi x + 3\,{}'''\mathrm{A}a^{m-3}\pi x^2 + \right\}$$
$$+ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

Having thus the forms of the series, nothing is required but to substitute them in the equation

$$(a+x)^m \times (a+y)^m = \left\{(a+x)(a+y)\right\}^m$$

and to compare the coefficients of the first power of y on each side, and we find

$$'\mathrm{A}a^{2m-1} + {}'\mathrm{A}\,{}'\mathrm{A}a^{2m-2}x + {}'\mathrm{A}\,{}''\mathrm{A}a^{2m-3}x^2 + {}'\mathrm{A}\,{}'''\mathrm{A}a^{2m-4}x^3 +$$
$$= {}'\mathrm{A}a^{2m-1}\pi + 2\,{}''\mathrm{A}a^{2m-2}\pi x + 3\,{}'''\mathrm{A}a^{2m-3}\pi x^2 + 4\,{}''''\mathrm{A}a^{2m-4}\pi x^3 +$$

$$=\left.\begin{matrix}'Aa^{2m-1}+2''Aa^{2m-2}\\ +\ 'Aa^{2m-2}\end{matrix}\right\}x\ \left.\begin{matrix}+3'''Aa^{2m-3}\\ +2''Aa^{2m-3}\end{matrix}\right\}x^2\ \left.\begin{matrix}+4''''Aa^{2m-4}\\ +3'''Aa^{2m-4}\end{matrix}\right\}x^3\ \begin{matrix}+\\ +\end{matrix}$$

by putting for π its value $1+\frac{x}{a}$. And by comparing the coefficients of the different powers of x there arise $'A='A$; $'A'A=2''A+'A$; $'A''A=3'''A+2''A$; $'A'''A=4''''A+3'''A$; and so on; whence $''A=\frac{'A('A-1)}{2}$; $'''A=\frac{''A('A-2)}{3}$; $''''A=\frac{'''A('A-3)}{4}$; and so on.

Such is the law by which the coefficients are derived from each other, whatever be the value of m; it remains to find $'A$; but I shall first observe, that if, instead of the assumed form of the expansion, we had made $(a+x)^m=a^m+'Aa^{m-1}x+''Ax^2+'''Ax^3+$ as some do, our demonstration would have succeeded exactly the same; because the exponent (m) and the first term (a) of the binomials are the same in all the three powers employed.

We have already seen that $'A=m$, if m be a whole positive number; or that $(a+x)^m=a^m+ma^{m-1}x+$; and from the value of $'A$ in this one case its value in all the others is easily discovered: thus, let

$(a+x)^{\frac{1}{m}}=a^{\frac{1}{m}}+'Aa^{\frac{1}{m}-1}x+$, the m^{th} power of this is $a+x$;

but $\left(a^{\frac{1}{m}}+'Aa^{\frac{1}{m}-1}x+\right)^m=a^{\frac{m}{m}}+m'Aa^{\frac{m-1}{m}}\times a^{\frac{1}{m}-1}x+=a+m'Ax+$

but $a+m'Ax+=a+x$, consequently $'A=\frac{1}{m}$; and

$(a+x)^{\frac{1}{m}}=a^{\frac{1}{m}}+\frac{1}{m}a^{\frac{1-m}{m}}x+$, next raise this to the n^{th} power, n being a positive integer,

$$(a+x)^{\frac{n}{m}}=\left(a^{\frac{1}{m}}+\frac{1}{m}a^{\frac{1-m}{m}}x+\right)^n=a^{\frac{n}{m}}+na^{\frac{n-1}{m}}\times\frac{1}{m}a^{\frac{1-m}{m}}x+$$

$= a^{\frac{n}{m}} + \frac{n}{m} a^{\frac{n}{m}-1} x +$. Lastly $(a+x)^{\frac{-n}{m}} = \frac{1}{(a+x)^{\frac{n}{m}}} =$

$= \frac{1}{a^{\frac{n}{m}} + \frac{n}{m} a^{\frac{n}{m}-1} x +} =$ (by actual division) $a^{\frac{-n}{m}} - \frac{n}{m} a^{\frac{-n}{m}-1} x +$

In all cases therefore 'A is equal to the exponent of the power.

XXV. *On the fluents of irrational functions.* *By* Edward Ffrench Bromhead, *Esq. M. A.* *Communicated by* J. F. W. Herschel, *Esq. F. R. S.*

Read June 4, 1816.

THE efforts of analysts in determining the fluents of rational functions, have been completely successful, and their labours form one of the most perfect and beautiful branches of the fluxionary calculus. In the irrational functions, however, we find but little effected. With the exception of WARING, modern analysts have not added any thing important, to the forms given by NEWTON, CRAIG, COTES, and BERNOULLI. No attempt has been made to generalize the known forms, and the last eminent writer on the subject, LA CROIX, seems to consider them as independent results, not deducible from any common principles, and refers us to the Petersburgh Acts, and other miscellaneous Collections. In the following pages, it is attempted to generalize and systematize our knowledge on this subject; and to show that all the known forms result from other forms of the greatest extent, not depending on particular functions, but upon the properties of all rational functions whatever.

$\mathrm{R}, \underset{1}{\mathrm{R}}, \underset{2}{\mathrm{R}}, \underset{n}{\mathrm{R}}$, denote rational functions of any kind; R^{-1}, $\underset{1}{\mathrm{R}}^{-1}$, $\underset{2}{\mathrm{R}}^{-1}$, $\underset{n}{\mathrm{R}}^{-1}$ their inverse functions. Thus if $x = \mathrm{R}(v)$ any rational function of (v), then $v = \mathrm{R}^{-1}(x)$ the inverse function.

It is thought unnecessary to prove, that the fluxions of all rational functions, and all rational functions of them, are themselves rational.

PROP. I.

$$dx \,.\, \underset{1}{\mathrm{R}}\left\{x, \underset{1}{\mathrm{R}}^{-1}(x)\right\} \text{ can be rationalized.}$$

Let $\underset{1}{\mathrm{R}}^{-1}(x) = v$; $x = \underset{1}{\mathrm{R}}(v)$; $dx = dv \,.\, \mathrm{D}\underset{1}{\mathrm{R}}(v)$*

which substituted produce the rational form

$$dv \,.\, \mathrm{D}\underset{1}{\mathrm{R}}(v) \,.\, \mathrm{R}\left\{\underset{1}{\mathrm{R}}(v), v\right\}$$

Cor. 1. This form includes

$$dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}(x), \underset{2}{\mathrm{R}}\left(x, \underset{1}{\mathrm{R}}^{-1}(x)\right), \ldots \underset{n}{\mathrm{R}}\left(x, \underset{1}{\mathrm{R}}^{-1}(x)\right)\right\}$$

Cor. 2. We may find, a priori, what fluents will come under this form. For let $x = \underset{1}{\mathrm{R}}(v)$ any rational function whatever

$$= \frac{\underset{0}{\alpha} + \underset{1}{\alpha} \,.\, v + \underset{2}{\alpha} \,.\, v^2 + \ldots}{\underset{0}{a} + \underset{1}{a} \,.\, v + \underset{2}{a} \,.\, v^2 + \ldots}$$

which is the general form taken by rational functions, when the integer powers are expanded, and the fractions reduced to a common denominator, the coefficients being positive, negative, or nothing. Hence we have $\underset{1}{\mathrm{R}}^{-1}(x) = v$, determined from this equation.

$$\left(\underset{0}{a} \,.\, x - \underset{0}{\alpha}\right) + \left(\underset{1}{a} \,.\, x - \underset{1}{\alpha}\right) . v + \left(\underset{2}{a} \,.\, x - \underset{2}{\alpha}\right) . v^2 + \ldots = 0$$

Cor. 3. Let the equation be

$$(ax - \alpha) + (bx - \beta)\, v^n = 0$$

Then $\underset{1}{\mathrm{R}}^{-1}(x) = v = \left\{\frac{-ax + \alpha}{bx - \beta}\right\}^{\frac{1}{n}}$

$\therefore$ we know $\int dx \,.\, \mathrm{R}\left\{x, \left(\frac{ax + \alpha}{bx - \beta}\right)^{\frac{1}{n}}\right\}$

* See Note at the end of the Paper.

Cor. 4. Let the equation be

$$(ax-\alpha)+(bx-\beta)\,.\,v^{n}+(cx-\gamma)\,.\,v^{2n}=0$$

from which we can determine

$$\int dx\,.\,\mathrm{R}\left\{x,\left(-\frac{bx-\beta}{2cx-2\gamma}\pm\sqrt{\left(\frac{bx-\beta}{2cx-2\gamma}\right)^{2}-\frac{ax-\alpha}{cx-\gamma}}\right)^{\frac{1}{n}}\right\}$$

and therefore any fluent of the form

$$\int dx\,.\,\mathrm{R}\left\{x,\sqrt[n]{\alpha x+\delta\pm\sqrt{a^{2}x^{2}+\beta x+\gamma}}\right\}$$

It is obvious that these deductions may be carried to any extent, producing forms hitherto supposed impracticable.

Cor. 5. $\frac{\alpha x+\beta}{ax+b}$ is both of the form $\mathrm{R}^{-1}(x)$ and $\mathrm{R}^{+1}(x)$.

PROP. II.

We can rationalize

$$dx\,.\,\mathrm{R}\left\{x,\underset{1}{\mathrm{R}}^{-1}(x);\underset{2}{\mathrm{R}}^{-1}\underset{1}{\mathrm{R}}^{-1}(x),\ldots\underset{n}{\mathrm{R}}^{-1}\underset{n-1}{\mathrm{R}}^{-1}\ldots\underset{2}{\mathrm{R}}^{-1}\underset{1}{\mathrm{R}}^{-1}(x)\right\}$$

Let $\underset{n}{\mathrm{R}}^{-1}\ldots\underset{1}{\mathrm{R}}^{-1}(x)=v.$ Then

$$x=\underset{1}{\mathrm{R}}\,\underset{2}{\mathrm{R}}\ldots\underset{n}{\mathrm{R}}(v)$$

$$\underset{1}{\mathrm{R}}^{-1}x=\underset{2}{\mathrm{R}}\ldots\underset{n}{\mathrm{R}}(v)$$

&c. = &c.

which substituted in the original expression, make it rational.

Cor. 1. If $\underset{1}{\mathrm{R}}=\underset{2}{\mathrm{R}}=\underset{n}{\mathrm{R}}$, the fluxion becomes

$$dx\,.\,\mathrm{R}\left\{x,\underset{1}{\mathrm{R}}^{-1}(x),\underset{1}{\mathrm{R}}^{-2}(x),\ldots\underset{1}{\mathrm{R}}^{-n}(x)\right\}$$

Cor. 2. By this theorem, any of the expressions deducible from Prop. I. may enter contemporaneously, and we may find fluents of very great intricacy.

Cor 3. The fluents

$$\int dx\,.\,\mathrm{R}\left\{x,(\alpha+\beta x)^{\frac{1}{n}},\left(\frac{a+b\sqrt[n]{\alpha+\beta x}}{c+d\sqrt[n]{\alpha+\beta x}}\right)^{\frac{1}{m}},\ldots\right\}$$

$$\int dx\,.\,\mathrm{R}\left\{x, \sqrt[n]{a+\sqrt[m]{b+\sqrt[r]{\ldots\sqrt[p]{\mathrm{R}^{-1}(x)}}}},\,.\,.\right\}$$

and some of the most complex expressions in WARING's Med. Anal. are very particular cases of this form.

PROP. III.

We can rationalize

$$dx\,.\,\mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}(x), \left(\underset{1}{\mathrm{R}}^{-1}(x)\right)^{\frac{1}{m}}, \left(\underset{1}{\mathrm{R}}^{-1}(x)\right)^{\frac{1}{n}}, \left(\underset{1}{\mathrm{R}}^{-1}(x)\right)^{\frac{1}{r}}, \ldots\right\}$$

$$\text{Let } \underset{1}{\mathrm{R}}^{-1}(x) = v^{m\,.\,n\,.\,r\,.\,.}$$

$$\text{Then } x = \underset{1}{\mathrm{R}}\,(v^{m\,.\,n\,.\,r\,.\,.\,.})$$

$$\left\{\underset{1}{\mathrm{R}}^{-1}(x)\right\}^{\frac{1}{m}} = v^{n\,.\,r\,.\,.}$$

$$\left\{\underset{1}{\mathrm{R}}^{-1}(x)\right\}^{\frac{1}{n}} = v^{m\,.\,r\,.\,.\,.}$$

$$\&c. \quad = \quad \&c.$$

which substituted make the expression rational.

Cor. 1. The more general form is this:

If $\underset{m}{\mathrm{R}}$ can be so assumed that $\underset{2}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}$, $\underset{3}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}$, $\underset{n}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}$ shall be all rational; then by assuming $\underset{1}{\mathrm{R}}^{-1}(x) = \underset{m}{\mathrm{R}}\,(v)$ we can render rational

$$dx.\,\mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}(x), \underset{2}{\mathrm{R}}^{-1}\,\underset{1}{\mathrm{R}}^{-1}(x), \ldots \underset{n}{\mathrm{R}}^{-1}\,\underset{1}{\mathrm{R}}^{-1}(x)\right\}$$

Cor. 2. We can find

$$\int dx\,.\,\mathrm{R}\left\{x,\ x^{\frac{1}{m}},\ x^{\frac{1}{n}}, x^{\frac{1}{r}}, \ldots\right\}$$

$$\int dx\,.\,\mathrm{R}\left\{x, (\alpha x+\beta)^{\frac{1}{m}}, (\alpha x+\beta)^{\frac{1}{n}}, \ldots\right\}$$

$$\int dx\,.\,\mathrm{R}\left\{x, \left(\frac{\alpha x+\beta}{ax+b}\right)^{\frac{1}{m}}, \left(\frac{\alpha x+\beta}{ax+b}\right)^{\frac{1}{n}}, \ldots\right\}$$

$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt{\alpha^2 x^2 + \beta x + \gamma}, (\alpha x + \sqrt{\alpha^2 x^2 + \beta x + \gamma})^{\frac{n}{m}}, \ldots\ldots\right\}$$

with an indefinite number of forms too complex for convenient expression.

Cor. 3. This form may be extended to Prop. II. and other general expressions. Thus we know

$$\int dx . \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}(x), \underset{2}{\mathrm{R}}^{-1}\underset{1}{\mathrm{R}}^{-1}(x), \ldots. (\underset{n}{\mathrm{R}}^{-1}..\underset{1}{\mathrm{R}}^{-1}(x))^{\frac{1}{m}}, (\underset{n}{\mathrm{R}}^{-1}...\underset{1}{\mathrm{R}}^{-1}(x))^{\frac{1}{n}}, \ldots\right\}$$

The forms given above are wholly inapplicable, when the fluxion involves expressions, such as $\underset{1}{\mathrm{R}}^{-1}\underset{2}{\mathrm{R}}\,\underset{3}{\mathrm{R}}^{-1}\ldots(x)$ where the functions are alternately inverse and direct. The cases are very few, in which the difficulty can be overcome, and perhaps the following Propositions will be found to include all the instances, in which analysts have effected the reduction.

Prop. IV.

We can rationalize

$$dx \,.\, \mathrm{D}\underset{2}{\mathrm{R}}(x) \,.\, \left\{\underset{2}{\mathrm{R}}(x), \underset{1}{\mathrm{R}}^{-1}\underset{2}{\mathrm{R}}(x)\right\}$$

Let $\underset{2}{\mathrm{R}}(x) = v$, and it becomes

$$dv \,.\, \mathrm{R}\left\{v, \underset{1}{\mathrm{R}}^{-1}(v)\right\} \text{ as in Prop. 1.}$$

Cor. 1. We can generally reduce $\int dx \,.\, \mathrm{D}\varphi(x) \,.\, \underset{1}{\varphi}\,\varphi(x)$ to $\int dv \,.\, \underset{1}{\varphi}(v)$. Thus we deduce from $\int dx \,.\, x^{rn-1} . \varphi(x^n)$ the $\int dv \,.\, v^{r-1}\varphi(v)$, and from $\int \frac{dx}{x} . \varphi(x^n)$ the $\int \frac{dv}{v} \,.\, \varphi(v)$, reductions of frequent occurrence, by which analysts have given their forms an appearance of generalization without the reality.

Cor. 2. This form may be extended to all the former Propositions.

Cor. 3. As it is very tedious and often impracticable to find x in terms of v, in order to know whether the reduction be applicable; the following process may sometimes be useful. Let the expression be

$$dx \,.\, \mathrm{D}\underset{2}{\mathrm{R}}(x) \,.\, \underset{1}{\mathrm{R}}^{-1}\underset{2}{\mathrm{R}}(x).\, \mathrm{R}\,\underset{2}{\mathrm{R}}(x)$$

Then if it be divided by $dx \,.\, \mathrm{D}\underset{2}{\mathrm{R}}(x) \,.\, \underset{1}{\mathrm{R}}^{-1}\underset{2}{\mathrm{R}}\ (x)$ the quotient will be a rational function of $\underset{2}{\mathrm{R}}\ (x)$ or of the form $\dfrac{\underset{0}{\alpha} + \underset{1}{\alpha}\,.\,\underset{2}{\mathrm{R}}(x) + \underset{2}{\alpha}\,.\,(\underset{2}{\mathrm{R}}\,(x))^2 + \,.\,.}{\underset{0}{a} + \underset{1}{a}\,.\,\underset{2}{\mathrm{R}}(x) + \underset{2}{a}\,.\,(\underset{2}{\mathrm{R}}\,(x))^2 + \,.\,.}$ the coefficients being indeterminate. If the reduction be applicable, these may be found, and the substitution made at once.

Cor. 4. We may thus reduce

$$\int dx \,.\, \frac{nx^{n-1} + (n-1)\,\underset{1}{c}\,.\,x^{n-2}}{\sqrt[m]{x^n + \underset{1}{c}x^{n-1} + \ldots}} \,.\, \mathrm{R}\left\{x^n + \underset{1}{c}\,x^{n-1} + \ldots\right\}$$

to $\int \frac{dv}{v^{\frac{1}{m}}}.\ \mathrm{R}\ (v)$, which may be found.

Cor. 5. In $dx \,.\, x^{rn-1}\,.(\alpha x^n + \beta)^{\mu}\,.\,\mathrm{R}\{\alpha x^n + \beta\}$ divide by $dx \,.\, n\alpha x^{n-1} \,.\, (\alpha x^n + \beta)^{\mu}$, and the quotient is $\frac{1}{n\alpha}.\, x^{(r-1)^n}.\,\mathrm{R}\,(\alpha x^n + \beta)$. The latter factor is already of the required form, and by assuming

$$x^{(r-1)^n} = \underset{0}{a}\,.\,(\alpha x^n + \beta)^{r-1} + \underset{1}{a}\,.\,(\alpha x^n + \beta)^{r-2} + \,.\,.$$

the indeterminates may be found. In particular cases there are readier processes, but this method is universally applicable.

Prop. V.

We can rationalize

$dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x)\right\}$ if we can determine, $\underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x) =$ $\underset{\nu}{\mathrm{R}}(x) \,.\, \underset{2}{\mathrm{R}}^{-1}(x)$, for the fluxion then becomes

$dx \,.\, \mathrm{R}\left\{x, \underset{\nu}{\mathrm{R}}(x) \,.\, \underset{2}{\mathrm{R}}^{-1}(x)\right\}$ as in Prop. I. Cor. 1.

Cor. 1. We also know

$$dx \,.\, \mathrm{R}\left\{x, \underset{2}{\mathrm{R}}^{-1}(x), \underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x)\right\}$$

Cor. 2. We may thus transform

$$dx \,.\, \mathrm{R}\left\{x, \left(\frac{\alpha x+\beta}{ax+b}\right)^{\frac{1}{m}}, \sqrt[n]{(\alpha x+\beta)\,.\,(ax+b)^{n-1}}\right\}$$

into

$$dx \,.\, \mathrm{R}\left\{x, \left(\frac{\alpha x+\beta}{ax+b}\right)^{\frac{1}{m}}, (ax+b)\,.\left(\frac{\alpha x+\beta}{ax+b}\right)^{\frac{1}{n}}\right\}$$

a known form.

Hence we know

$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt[3]{(\alpha^2 x^2 - \beta^2)\,.\,(\alpha x \pm \beta)}\right\}$$
$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt[3]{(x+\alpha)\,.\,(x+\beta)^2}\right\}$$
$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt{(\alpha x+\beta)\,.\,(ax+b)}\right\}$$
$$\int dx \,.\, \mathrm{R}\left\{x, \sqrt{(cx^2 + \underset{1}{c}\,x + \underset{2}{c})}\right\}$$

which last form will sometimes introduce imaginaries, that may be avoided by particular artifices.

Cor. 3. If $\underset{1}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}(x) = \left(\underset{\mu}{\mathrm{R}}(x)\right)^m \,.\, \underset{2}{\mathrm{R}}^{-1}(x)$

$$\underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x) = \left\{\underset{\nu}{\mathrm{R}}(x)\right\}^n \,.\, \underset{2}{\mathrm{R}}^{-1}(x)$$

&c. = &c.

we can determine

$$\int dx \,.\, \mathrm{R}\left\{x, \underset{2}{\mathrm{R}}^{-1}(x), \left\{\underset{1}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}(x)\right\}^{\frac{1}{m}}, \left\{\underset{2}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x)\right\}^{\frac{1}{n}}, \ldots\right\}$$

Cor. 4. If $\underset{1}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}(x) = \underset{\mu}{\mathrm{R}}(x)\,{}^{m}\sqrt{\underset{\alpha}{\mathrm{R}}^{-1}(x)}$; $\underset{2}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x) = \underset{\nu}{\mathrm{R}}(x)\,.$ ${}^{n}\sqrt{\underset{\alpha}{\mathrm{R}}^{-1}(x)}$; &c. = &c.,

we know

$$\int dx \,.\, \mathrm{R}\left\{x, \underset{\alpha}{\mathrm{R}}^{-1}(x), \underset{1}{\mathrm{R}}^{-1}\underset{m}{\mathrm{R}}(x), \underset{2}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(x), \underset{3}{\mathrm{R}}^{-1}\underset{r}{\mathrm{R}}(x), \ldots\right\}$$

as in

$$dx \,.\, \mathrm{R}\left\{x, \sqrt[m]{(\alpha x+\beta).(ax+b)^{m-1}}, \sqrt[n]{(\alpha x+\beta).(ax+b)^{n-1}}, \ldots\right\}$$

Cor. 5. Generally if $\underset{1}{\mathrm{R}}(x)$ and $\underset{2}{\mathrm{R}}(x)$ are so related, that $\underset{1}{\mathrm{R}}\,\underset{n}{\mathrm{R}}(x)$ can $= \underset{2}{\mathrm{R}}\,\underset{m}{\mathrm{R}}(x)$, $\underset{n}{\mathrm{R}}$ and $\underset{m}{\mathrm{R}}$ being any rational functions whatever taken at pleasure, then $dx \,.\, \mathrm{R}\left\{x, \underset{1}{\mathrm{R}}^{-1}\underset{2}{\mathrm{R}}(x)\right\}$ can be rationalized by taking $x = \underset{m}{\mathrm{R}}(v)$. It then becomes

$$dv \,.\, \mathrm{D}\underset{m}{\mathrm{R}}(v) \,.\, \mathrm{R}\left\{\underset{m}{\mathrm{R}}(v), \underset{n}{\mathrm{R}}(v)\right\}.$$

PROP. VI.

By combining Prop. IV. and V, we can rationalize

$$dx \,.\, \mathrm{D}\underset{m}{\mathrm{R}}(x) \,.\, \mathrm{R}\left\{\underset{m}{\mathrm{R}}(x), \underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}\,\underset{m}{\mathrm{R}}(x)\right\}$$

if $\underset{1}{\mathrm{R}}^{-1}\underset{n}{\mathrm{R}}(v) = \underset{\nu}{\mathrm{R}}(v) \,.\, \underset{2}{\mathrm{R}}^{-1}(v)$; for let $\underset{m}{\mathrm{R}}(x) = v$, and it becomes

$$dv \,.\, \mathrm{R}\left\{v, \underset{\nu}{\mathrm{R}}(v) \,.\, \underset{2}{\mathrm{R}}^{-1}(v)\right\} \text{ as before.}$$

Cor. 1. If we have

$$dx \,.\, x^{m-1}(\alpha x^{n} + \beta)^{\frac{p}{q}} \,.\, \mathrm{R}\left\{\alpha + \beta x^{-n}\right\}$$

Remove the multiplier x^n, as in Prop. V., and it becomes,

$$\frac{dx}{x} \cdot x^{m+\frac{np}{q}} \cdot (\alpha + \beta x^{-n})^{\frac{p}{q}} \cdot \mathrm{R}\,(\alpha + \beta x^{-n})$$

which will fall under Prop. IV. If $x^{m+\frac{np}{q}}$ can be expressed by a rational function of $\alpha + \beta x^{-n}$. This will happen if $m + \frac{np}{q} = -n \cdot r$, or if $\frac{m}{n} + \frac{p}{q} = \pm r$ any integer. Hence we know

$$\int \frac{dx}{(ax^{\alpha} + bx^{\beta})^{\frac{1}{\alpha}}} \cdot \mathrm{R}\left\{a + bx^{\beta-\alpha}\right\}$$

$$\int \frac{dx}{\sqrt[n]{\alpha + \beta x^n}}; \int \frac{dx}{\sqrt[3]{x^3 + 1}}; \int \frac{dx}{\sqrt[4]{x^4 - 1}};$$

Cor. 2. We can determine

$$\int \frac{\alpha x^{2n} - \beta}{\alpha x^{2n} + \gamma x^n + \beta} \cdot \frac{dx}{\sqrt[n]{\alpha x^{2n} + cx^n + \beta}}$$

which becomes by Prop. 5.

$$\frac{\alpha x^{n-} - \beta x^{-n-1}}{\alpha x^n + \beta x^{-n} + \gamma} \times \cdot \frac{dx}{\sqrt[n]{\alpha x^n + \beta x^{-n} + c}}$$

Now this falls under Prop. IV. For let

$$\sqrt[n]{\alpha x^n + \beta x^{-n} + c} = v$$

$$\alpha x^n + \beta x^{-n} + \gamma = v^n + \gamma - c$$

$$\therefore (\alpha x^{n-1} - \beta x^{-n-1}) \cdot dx = v^{n-1} \cdot dv$$

Whence the fluxion becomes $\frac{v^{n-2} \cdot dv}{v^n + \gamma - c}$, of which a particular case is deduced in Legendre's Elliptic Transcendents.

Prop. VII.

If we can rationalize

$$dx \cdot \mathrm{R}\left\{x, \varphi\,(x), \varphi_{2}\,(x), \ldots\right\}$$

we also can

$$dx \,.\, R\left\{x, \underset{1}{R}^{-1}(x), \varphi \underset{1}{R}^{-1}(x), \underset{2}{\varphi} \underset{1}{R}^{-1}(x), \ldots\right\}$$

for by taking $\underset{1}{R}^{-1}(x) = v$, it is reduced to the former form.

Cor. 1. If we can rationalize

$$dx \,.\, R\left\{x, \underset{1}{R}^{-1} \underset{n}{R}(x)\right\}, \text{ we also can}$$

$$dx \,.\, R\left\{x, \underset{m}{R}^{-1}(x), \underset{1}{R}^{-1} \underset{n}{R} \underset{m}{R}^{-1}(x)\right\}$$

Therefore we can find

$$\int dx \,.\, R\left\{x, \underset{m}{R}^{-1}(x), \sqrt{a + b \,.\, \underset{m}{R}^{-1}(x) + c \,.\, \left(\underset{m}{R}^{-1}(x)\right)^2}\right\}$$

$$\int dx \,.\, R\left\{x, \sqrt{\alpha + \beta x + \sqrt{a + bx}}\right\} \text{ \&c.}$$

Cor. 2. Generally we can reduce

$$dx \,.\, R\left\{x, \underset{0}{R} \underset{1}{R}^{-1} \ldots \underset{n}{R} \underset{n+1}{R}^{-1}(x)\right\} \text{ to}$$

$$dx \,.\, R\left\{x, \underset{1}{R}^{-1} \ldots \underset{n}{R}(x)\right\}$$

Cor. 3. In $\int dx \,.\, \varphi(x)$, let $v = \varphi(x)$, and if it be an algebraic function, $R\left\{x, v,\right\} = o$. Now take $x = \underset{1}{R}(z)$ $= \dfrac{\alpha + \underset{1}{\alpha} z + \underset{2}{\alpha} z^2 + \&c.}{a + \underset{1}{a} z + \underset{2}{a} z^2 + \&c.}$; and $v = \underset{2}{R}(z) = \dfrac{\beta + \underset{1}{\beta} z + \&c.}{b + \underset{1}{b} z + \&c.}$ with indeterminate coefficients.

Hence we have $R\left\{\underset{1}{R}(z), \underset{2}{R}(z)\right\} = o$; remove fractions and make the coefficients of the powers of z vanish. This will give the indeterminates, if x and v admit common rationalities. Thence we have $dz \,.\, D\underset{1}{R}(z) \,.\, \underset{2}{R}(z)$ rational.

Should all the artifices in the foregoing propositions fail, we must attempt to resolve the fluxion into a series of terms, such that each term may be separately rationalized. This is

often possible, when the original function does not admit a rational expression, and can be effected sometimes directly, and sometimes by introducing a new variable. But it will first be necessary to reduce all irrational functions whatever to a definite form.

LEMMA.

To reduce all irrational functions to a definite form.

1. By successively multiplying numerators and denominators into the same expressions, every irrational function may at last be reduced to a series of terms, whose numerators and denominators do not contain any fraction or negative index.

Thus $\frac{\left(\alpha + \frac{\beta}{\gamma}\right)^{\mu}}{(a + bc^{-n})^m} = \frac{c^{nm} \cdot (\alpha\gamma + \beta)^{\mu}}{\gamma^{\mu} \cdot (ac^n + b)^m}$, and if $\alpha, \beta, \gamma, a, b, c$, are functions involving fractions or negative indices, themselves, the reduction is continued in the same manner.

2. Now multiply both the numerators and denominators of the expressions so reduced, by such multipliers, as will render the denominators rational. This factor is the product of all the different values of the denominator, with the exception of the denominator itself. The new numerators will still consist of a series of terms not involving any fraction or negative index.

3. If R, R_1, R_2, &c. denote functions of the form $cx^m + c_1 \cdot x^{m-1} + \ldots$, the irrational takes the form,

$$\frac{R(x)}{R_1(x)} + \frac{R_2(x)}{R_3(x)} \cdot \phi_1(x)^{\frac{a}{b}} \cdot \phi_2(x)^{\frac{c}{d}} \ldots + \frac{R_4(x)}{R_5(x)} \cdot \pi_1(x)^{\frac{\alpha}{\beta}} \ldots + \ldots$$

4. By reducing the fractional indices of the factors to the

common denominator (n), the whole will consist of a series of terms $\frac{\underset{2}{R}(x)}{\underset{3}{R}(x)} \cdot \sqrt[n]{\underset{1}{\varphi}(x)^p \cdot \underset{2}{\varphi}(x)^q \ldots}$

5. By expanding all the integer powers under the index $\frac{1}{n}$; and again reducing the indices of the sums and products, which are under it, to a common denominator n'; we shall by continuing the same operations, ultimately reduce the whole expression, to a series of terms of the form

$$\frac{\underset{2}{R}(x)}{\underset{3}{R}(x)} \sqrt[n]{S^{n'}\sqrt{S^{n''}\sqrt{\ldots S^{n''\ldots}\sqrt{\underset{m}{R}(x)}}}}$$

S denoting the sum of any number of terms such as follow it, wherein $\underset{m}{R}(x)$ may be different in each term, but always of the form $cx^m + \underset{1}{c}x^{m-1} + \ldots +$.

6. If *every* value of $\underset{m}{R}(x)$ contains a factor $(ax^r + bx^{r-1} + \ldots)^{n \cdot n' \cdot n'' \ldots}$, it may be taken entirely out of the radical; and conversely the rational coefficient may be introduced entirely under the radical.

7. When the surd is so reduced, that no rational factor can be withdrawn from the radical, it is said to be in its lowest terms; and is said to be an irrational of the 1st, 2d, or ν^{th} order, according to the number of the indices $\frac{1}{n}$, $\frac{1}{n'}$, $\frac{1}{n^{\prime\prime\ldots\nu-1}}$. Thus the general expression for a surd of the first order

is a series of terms, $\frac{\underset{2}{R}(x)}{\underset{3}{R}(x)} \sqrt[n]{cx^m + \underset{1}{c}x^{m-1} + \ldots}$

8. A more convenient general form for all irrationals, than

the series of terms above exhibited, may readily be found; by introducing all the rational parts entirely under the radicals; by reducing the indices of all the terms to a common denominator μ; by expanding all integer powers; and by again reducing all the products and sums contained under $\frac{1}{\mu}$, to indices with a common denominator μ'. These operations continued, will ultimately lead to the expression

$S\sqrt[\mu]{S\sqrt[\mu']{\ldots S\sqrt[\mu'\ldots]{\frac{\underset{\alpha}{R}(x)}{\underset{\beta}{R}(x)}}}}$, where $\frac{\underset{\alpha}{R}(x)}{\underset{\beta}{R}(x)}$ may be of any different values in the different sums, but always of the form $\frac{ax^{\alpha} + \underset{1}{a}x^{\alpha-1} + \ldots}{bx^{\beta} + \underset{1}{b}x^{\beta-1} + \ldots}$

9. $\frac{\underset{\alpha}{R}(x)}{\underset{\beta}{R}(x)}$ is said to be of $\alpha - \beta$ dimensions; and if $\alpha - \beta$ be dimensions of that rational part, whose dimensions are greatest; then the dimensions of the whole irrational are $\frac{\alpha-\beta}{\mu\,.\,\mu'\,.\,\mu''\ldots}$.

10. The fluxion, and its dimensions in any irrational, may be found by applying this formula, $d\left\{\underset{1}{\varphi}\,\underset{2}{\varphi}\,\underset{3}{\varphi}\ldots\underset{n}{\varphi}(x)\right\} = D\,\underset{1}{\varphi}\,\underset{2}{\varphi}\ldots\underset{n}{\varphi}(x)\,.\,D\,\underset{2}{\varphi}\,\underset{3}{\varphi}\ldots\underset{n}{\varphi}(x)\ldots D\,\underset{n}{\varphi}(x)\,.\,dx$, the D only referring to the functional characteristic immediately succeeding it.

Prop. VIII.

To divide a fluxion into expressions admitting distinct rationalities.

Let $\varphi\ (x)$ be any irrational, and $\underset{1}{\varphi}\ (x)$, $\underset{2}{\varphi}\ (x)$ &c. surds deduced as in the Lemma. Then

$$\int dx\ .\ \varphi\,(x) = \int \frac{dx\ .\ \mathrm{R}\ x)}{\underset{1}{\mathrm{R}}\,(x)} + \int \frac{dx\ .\ \underset{2}{\mathrm{R}}\,(x)}{\underset{3}{\mathrm{R}}\,(x)}\ .\ \underset{1}{\varphi}\,(x) + \&c.$$

where the fluent of the 1st term may always be found, and the other terms may often be rationalized by distinct substitutions, when we are unsuccessful with $\int dx.\ .\ \varphi\ (x)$. Again since in each of the terms,

$\int \frac{dx\ .\ \underset{2}{\mathrm{R}}\,(x)}{\underset{3}{\mathrm{R}}\,(x)}\ .\ \underset{1}{\varphi}\ (x)$, $\frac{\underset{3}{\mathrm{R}}\ (x)}{\underset{3}{\mathrm{R}}\ (x)}$ may be reduced to a series of terms of the form $\mathrm{A}x^n$ and $\frac{\mathrm{A}}{(x\ +\ a)^n}$; therefore the fluent depends on a series of terms $\int dx\ .\ x^n\ .\ \underset{1}{\varphi}\ (x)$, and $\int dx\ .\ (x+a)^{-n}\ .\ \underset{1}{\varphi}\ (x)$. In the latter case, the form of $\underset{1}{\varphi}\ (x)$ is not changed by substituting x for $x\ +\ a$, and $\therefore$ the fluents of all irrationals are determinable by $\int dx\ .\ x^{\pm n}\ .\ \underset{1}{\varphi}\ (x)$.

Cor. 1. If we multiply the denominator of

$$\frac{dx\ .\ \mathrm{R}\ (x)}{\underset{1}{\mathrm{R}}\,(x)\ .\ \sqrt{ax^2 + bx + c} + \underset{2}{\mathrm{R}}\,(x)\ .\ \sqrt{\alpha x^2 + \beta x + \gamma}}$$

by its rationalizing factor, the fluxion will be reduced to two terms, which admit distinct rationalities.

Cor. 2. Sometimes by the substitution of a new variable, for some function of x, the fluxion will be divided into a series of terms, each of which may be separately made rational. But unfortunately no general principle has been discovered, to which these reductions can be referred.

Cor. 3. As the fluent of each term can sometimes be found

apart, when the fluent of the whole cannot be found at once; so conversely, the fluent of a series of terms may be found, when each separate term surpasses the powers of analysis. Thus we know

$$\int \frac{d\varphi(x) + d\underset{1}{\varphi}(x) + \ldots}{\varphi(x) + \underset{1}{\varphi}(x) + \ldots}$$

But we do not know

$$\int \frac{d\varphi(x)}{\varphi(x) + \underset{1}{\varphi}(x) + \ldots} + \int \frac{d\underset{1}{\varphi}(x)}{\varphi(x) + \underset{1}{\varphi}(x) + \ldots} + \ldots$$

Again, let $\varphi(x)$ be such a function of x,

that $\varphi^2(x) = \frac{x}{e}$; let $\varphi(x) = \underset{1}{x}$;

$$\begin{aligned} \text{Then} \int dx \,.\, \varphi(x) &= \underset{1}{x} \,.\, \varphi(x) - \int d\varphi(x) \,.\, x \\ &= x \,.\, \varphi(x) - \int d\varphi(x) \,.\, \varphi^2(x) \,.\, e \\ &= x \,.\, \underset{1}{x} \quad - e \int d\underset{1}{x} \,.\, \underset{1}{\varphi}(\underset{1}{x}) \end{aligned}$$

$$\therefore \int dx \,.\, \varphi(x) + e \int d\underset{1}{x} \,.\, \varphi(\underset{1}{x}) = x \,.\, \underset{1}{x}$$

Which theorem admits farther extension, and may be applied to elliptic arches.

Should the above processes for rendering the fluxion rational fail us, we must attempt the fluxion at once in its irrational state, for which purpose I shall add a few miscellaneous observations.

1. If $\varphi(x)$, $\underset{1}{\varphi}(x)$ be any algebraic functions,

then $d\left\{\varphi(x) + \log. \underset{1}{\varphi}(x)\right\} = d\varphi(x) + \frac{d\underset{1}{\varphi}(x)}{\underset{1}{\varphi}(x)}$ is an algebraic expression. Whenever, therefore, we meet with an

algebraic fluxion, we may legitimately try $\varphi(x) + \log. \underset{1}{\varphi}(x)$, as a form to which the fluent may possibly belong.*

2. It presents three cases: 1st. where the fluent is wholly algebraic, for which we assume some expression so general, that its fluxion will include the given fluxion, if it admit an algebraic fluent; or we find the fluent implicitly by an equation: 2dly. where the fluent is mixed, when we attempt to separate the algebraic part: 3dly. where the fluent is purely logarithmic, when we assume, as in the first case, some expression with indeterminate constants, sufficiently general to include the given fluxion.

3. In assuming for an algebraic fluxion, it must be observed, that the fluent will always be a surd of the same order as the fluxion. On this principle WARING gives some assumptions for surds of the second order, but nothing has been attempted generally for surds of all orders, for want of some definite form which should include them all. In irrationals of the first order, the fluxion may always be reduced to series of terms, such as

$$dx \cdot \left(x + \underset{1}{a}\right)^{\alpha} \cdot \left(x + \underset{2}{a}\right)^{\beta} \cdots \left(x + \underset{n}{a}\right)$$

where the factors are all different, and where the indices are positive, negative, fractions, integers, or unity. Then let $\underset{n-1}{R}(x)$ be any expression $cx^{n-1} + \underset{1}{c}x^{n-2} + \ldots$ with indeterminate coefficients. Assume for the fluent

$$\frac{\left(x + \underset{1}{a}\right)^{\alpha+1} \cdot \left(x + \underset{2}{a}\right)^{\beta+1} \cdots \left(x + \underset{n}{a}\right)^{\nu+1}}{\underset{n-1}{R}(x)}$$

* It is obvious, that the fluent of an algebraic fluxion cannot be of the form $\varphi(x) + \underset{1}{\varphi}(x) \cdot \log. \underset{2}{\varphi}(x)$, for its fluxion $d\varphi(x) + d\underset{1}{\varphi}(x) \cdot \log. \underset{2}{\varphi}(x) + \underset{1}{\varphi}x \cdot \dfrac{d\underset{2}{\varphi}(x)}{\underset{2}{\varphi}(x)}$ is a transcendent.

Its fluxion will be

$$\frac{DR_{n-1}(x)\cdot\left(x+\underset{1}{a}\right)\cdot\left(x+\underset{2}{a}\right)\cdot\cdot\left(x+\underset{n}{a}\right)}{\left\{R_{n-1}(x)\right\}^2}$$

$$\frac{-R_{n-1}(x)\cdot\left\{(\alpha+1)\cdot\left(x+\underset{2}{\alpha}\right)\cdot\cdot\left(x+\underset{n}{a}\right)+(\beta+1)\cdot\left(x+\underset{1}{a}\right)\cdot\cdot\left(x+\underset{n}{a}\right)+\cdot\right\}}{\left\{R_{n-1}\,x)\right\}^2}$$

multiplied by $dx\,.\,(x+\underset{1}{a})^{\alpha}\,.\,.\,.\,(x+\underset{n}{a})^{\nu}$ the original fluxion.

Now that the two expressions may be equal, the coefficient found above must be $=1$, or we must have

$$DR_{n-1}(x)\,.\,Q-R_{n-1}(x)\,.\,Q'=\left\{R_{n-1}(x)\right\}^2$$

where Q and Q′ are the expressions in the coefficient involving $\underset{1}{a}, \underset{2}{a}, \underset{n}{a}$. By equating the terms in this equation, the indeterminates $c, \underset{1}{c}, \underset{2}{c}$, &c. may be found; but the reduction will often be impossible, as there are more equations to be satisfied than there are indeterminates.

4. If any index $\alpha, \beta, \gamma = -1$, the expression fails, and there is no algebraic fluent; also Waring says, that if the dimensions of a fluxional coefficient be $= -1$, the fluent cannot be algebraic.

5. If $\varphi\,(x)$ be an irrational function, let $z=\int dx\,.\,\varphi\,(x) = \int dx\,.\,v$; then since $R\left\{x, v\right\}=0$, there are cases, where we can determine, $R\left\{x, z\right\}=0$.*

6. If $\varphi\,(x), \pi\,(x)$ be irrational functions of x, we have

$$\int dx\,.\,\varphi\,(x)=\int dx\,.\,\left\{\varphi\,(x)+\pi\,(x)\right\}-\int dx\,.\,\pi\,(x)$$

Now let $\pi\,(x)$ be so assumed, that

* See Phil. Trans. 1764.—Emerson's Fluxions.

$\int dx \cdot \left\{\varphi(x) + \pi(x)\right\} = \underset{1}{\varphi}(x)$, and we have

$$\int dx \cdot \varphi(x) = \underset{1}{\varphi}(x) - \int dx \cdot \pi(x).$$

If therefore $\pi(x)$ be a simpler expression than $\varphi(x)$, the fluxion will be reduced to a simpler form. In order to find $\pi(x)$; $\underset{1}{\varphi}(x)$ is assumed with indeterminate coefficients, so that its fluxion may be of the same form as $\dot{dx} \cdot \varphi(x)$. Now equate the similar terms in the two expressions, and the indeterminates may be found. But as there may be more equations than indeterminates, we add $\pi(x)$ a function of the same form, and containing indeterminates of sufficient number, to satisfy all the deficient equations. We shall thus have $D\underset{1}{\varphi}(x) = \varphi(x) + \pi(x)$ and $\therefore \underset{1}{\varphi}(x) = \int dx. \left\{\varphi(x) + \pi(x)\right\}$ by which the difficulty may be reduced to finding $\int dx \cdot \pi(x)$. Reductions of particular kinds were discovered by Simpson and others, but this is universally applicable.

7. It may be of advantage to reduce the index of the variable under the radical, which may sometimes be effected. In $\dot{dx} \cdot \left\{x^{m+n} + 1\right\}^{\frac{1}{n}}$, assume $x^{m+n} + 1 = v^n$;

Then we have

$$\dot{dx} \cdot \left\{x^{m+n} + 1\right\}^{\frac{1}{n}} = \frac{n}{m+n} \cdot \dot{dv} \cdot v^n \cdot (v^n - 1)^{\frac{1}{m+n} - 1}$$

And in the same manner surds of one order may be transformed into another.

8. If the fluent be wholly logarithmic, we may assume for irrationals of the first order

$$\log. \left\{\underset{0}{R}(x) + \underset{1}{R}(x) \cdot (x+a)^{\alpha} \cdot (x+b)^{\beta} \ldots + \underset{2}{R}(x) \cdot (x+c)^{\gamma} \ldots + \right\}$$

9. I shall conclude by observing that the fluxion may always be made rational, if the fluent be wholly algebraic, or wholly logarithmic. Thus, if $\varphi\,(x)$ be any algebraic function, take $x = \varphi^{-1}\,\mathrm{R}\,(v)$,

Then $d\varphi\,(x) \quad = d\left(\varphi\varphi^{-1}\,\mathrm{R}\,(v)\right) = d\,\mathrm{R}\,(v)$

and $d\left(\log.\,\varphi(x)\right) = d\left(\log.\,\varphi\varphi^{-1}\,\mathrm{R}(v)\right) = \frac{d\,\mathrm{R}\,(v)}{\mathrm{R}\,(v)}$

are both rational. If the fluent be of the mixed form $\varphi\,(x) + \log.\,\varphi_1\,(x)$, its fluxion may be made rational, if R, R_1, can be so assumed that $\varphi^{-1}\,\mathrm{R}\,(x) = \varphi_1^{-1}\,\mathrm{R}_1\,(x)$; and it may always be effected by introducing two new variables.

First let $x = \varphi^{-1}\,\mathrm{R}\,(v)$, and the fluxion becomes

$d\,\mathrm{R}\,(v) + \frac{d\left(\varphi_1\varphi^{-1}\mathrm{R}\,(v)\right)}{\varphi_1\varphi^{-1}\mathrm{R}\,(v)}$; now let $v = \mathrm{R}^{-1}\,\varphi\varphi_1^{-1}\,\mathrm{R}_1\,(z)$, an

we get $d\,\mathrm{R}\,(v) + \frac{d\,\mathrm{R}_1\,(z)}{\mathrm{R}_1\,(z)}$ which is wholly rational.

EDWARD FFRENCH BROMHEAD.

Note.—As modern analysts have in general confounded the fluxions, either with the increments or the derived functions, it may not be superfluous to state precisely, what is meant by the symbols d and D.

If it be possible, (which must be shown in each particular case) to expand $\varphi\,(x+v)$ in the form $\varphi\,(x) + \varphi_1\,(x)\,. \quad + \varphi_2\,(x)\,.\,v^2 + \ldots$; then $\varphi_1\,(x)$ is called the derived function of $\varphi\,(x)$, and its relation to $\varphi\,(x)$ is thus expressed, $\varphi_1\,(x) = \mathrm{D}\varphi\,(x)$. Hence, if x be considered a function of itself, we have $(x+v) = (x) + \mathrm{D}\,(x)\,.\,v$, and $\therefore$ $\mathrm{D}\,(x) = 1$.

Now to avoid a constant reference to the variable x, of which other variables are considered as functions, we introduce fluxions. If $y, z, w, \ldots$ are functions of the

same variable, then dy, dz, dw, . . are expressions *proportional* to the derived functions of y, z, w, . . whatever may be the variable of which they are common functions. Hence $\frac{dy}{dz} = \frac{Dy}{Dz}$; and if y be a function of x, or $= \varphi(x)$, then $\frac{dy}{dx} = \frac{D\varphi(x)}{Dx} = D\varphi(x)$ and $\therefore dy = dx \,.\, D\varphi(x)$.

Moreover, since the derived functions are in the limiting ratio of the increments, so also are the fluxions. From this consideration we can in the applications of analysis, *practically* determine the ratio of the fluxions, when the derived functions are unknown.

ERRATA.

Page 72, line 20, for *parts*, read *part*.
—— 73, line 3, for *between*, read *below*.
—— 98, line 4 from bottom, dele the comma after A.
—— 101, line 6 from bottom, dele BH.
—— 102, line 4, for *axes*, read *axis*.
—— 164, line 11, dele the comma between m and n.
—— 174, line 7, for *consisted of*, read *consisted in*.
—— ——, line last, for m, n, read m, m.
—— 191, line 13, for $\varphi\overset{-1}{\varphi} x$, read $\overset{-1}{\varphi} \varphi x$.
—— 213, line 14, for $\psi^{p} \psi(x, y)$, read $\varphi^{p} \psi(x, y)$.
—— 214, line 10, dele "*in an infinite number of ways*".
—— 224, line 22, for $f(a)$, read $f(x)$.
—— 226, line 24, for $= x$, read $= z$.
—— 232, line 16, *in the denominator*, for $1 -$, read $1 +$.
—— ——, line 18, *ditto*, *ditto*, for $1 -$, *read* $1 \pm$.
—— 251, line 9, for $\frac{d\psi\, x, \frac{1}{y})}{dx}$ read $\frac{d\psi\,(x, \frac{1}{y})}{dx}$
—— ——, line 11, for d *in both numerator*, read d^2.
—— —— line 13, for $\left(\frac{x}{y}\right)$ read $x\,\varphi\left(\frac{x}{y}\right)$.

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	Annals of Philosophy, Nov. 39.	Dr. Thomas Thomson.
Mar. 4.	Vetusta Monumenta, Vol. IV. Plates 36 to 52.	The Society of Antiquaries.
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May 2.	Annals of Philosophy, No. 41.	Dr. Thomas Thomson.
	The European Magazine for April.	The Editor.
	The Monthly Review for April.	Mr. G. E. Griffiths.
	The Philosophical Magazine for April.	Mr. Alexander Tilloch.
9.	The Principles of Population and Production, by John Weyland, Jun. Esq. London, 1816. 8°	John Weyland, Esq.
16.	Catalogue raisonné des Diamants dans le Cabinet de Sir Abraham Hume, Bart. par M. Le Comte de Bournon. Londres, 1815.	Sir Abraham Hume, Bart.
30.	Annals of Philosophy, No. 42.	Dr. Thomas Thomson.
June 13.	Journal des Mines, No. 219 and 220.	Ecole des Mines.
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	The Philosophical Magazine for May.	Mr. Alexander Tilloch.
	The European Magazine for May.	The Editor.
	The Monthly Review for May, and Appendix to Vol. LXXIX.	Mr. G. E. Griffiths.
July 4.	Leçons de Mechanique analytique, Tome II. par M. Prony.	M. Prony.
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